

Signatures and D-spectra and Their Use in Reliability Calculations: A Review

ILYA B. GERTSBAKH¹ and YOSEPH SHPUNGIN²

¹ Department of Mathematics, Ben-Gurion University
 P. O. Box 653, Beer-Sheva, 84105, Israel

² Software Engineering Department
 Sami Shamoon College of Engineering, Beer Sheva 84100 Israel

(Received on March 06, 2011 and revised on July 22, 2011 and May 19, 2012)

Abstract: Coherent systems with binary components have important structural parameters known in literature as signatures and D-spectra. The knowledge of these parameters allows to obtain the probabilistic description of coherent systems behavior in the process of their component failures, and such system characteristics as probabilistic resilience, component importance, system failure probability as a function of component failure probability q , approximations to system reliability, and the bounds on system reliability if we know only the bounds on q .

When the system has many components, the exact calculation of signatures or D-spectra becomes a very complicated issue. We suggest to use for their approximation efficient Monte Carlo procedures. All relevant calculations are illustrated by examples of networks.

This paper provides a nonformal review of relevant literature and the methodology of reliability calculations, based mostly on recently published books and research papers.

Keywords: Coherent systems, signature, D-spectra, component importance, Monte Carlo, network terminal reliability, approximations to reliability, bounds .

1 Monotone System: Definition

The purpose of this review is to demonstrate how D-spectra and signatures can be used in system reliability calculations.

We will consider in this paper monotone systems with three states and binary components. Component i state is denoted by a binary variable $x_i = 1(0)$ where 1 indicates that component is *up* and 0 - that it is *down*. The system is determined by a *structure function*

$$\varphi(\mathbf{x}) \rightarrow \{UP, DOWN1, DOWN0\},$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n)$ is the component state vector, *UP* and *DOWN0* are the states where the system is completely operable or failed, respectively, and *DOWN1* is an intermediate state of partial failure. As in standard definitions of monotone systems [2,9], it will be assumed that $\varphi(\mathbf{1}) = UP, \varphi(\mathbf{0}) = DOWN0$. Assume also that if one of the components fails, the system either remains in the same state or moves one step down, i.e., $UP \rightarrow DOWN1$ or $DOWN1 \rightarrow DOWN0$. It is desirable to have a visual image of a monotone system. For this purpose, it is convenient to think of a monotone system as of a network. Network \mathbf{N} is a collection of *nodes* V , and *edges (links)* E connecting the nodes. A subset of nodes $T \subseteq V$ have special status and we call them *terminals*. Thus, the network is a triple:

$$\mathbf{N} = (V, T, E), |V| = m, |E| = n.$$

If not said otherwise, the components subject to failure are the links. (Generally, it is possible to consider the case when the nodes are subject to failures).

Standard images of monotone systems appearing in each textbook are series-parallel systems, which in fact also can be represented as networks with two terminals. The well-known *bridge* is a network with four nodes s, a, b, t , five edges $1 = (s, a)$, $2 = (s, b)$, $3 = (a, b)$, $4 = (a, t)$, $5 = (b, t)$ and two terminals $T = (s, t)$. This network has, by definition, only two states: *UP* and *DOWN0*: *UP* state means that s and t are connected by a chain of links being in *up* state. More interesting situations arise when the number of terminals exceeds 2.

Example 1- three-dimensional cube.

Let us consider a three - dimensional cubic network H_3 having 12 edges, 8 nodes and 3 terminals (bold), shown on Fig. 1,a.

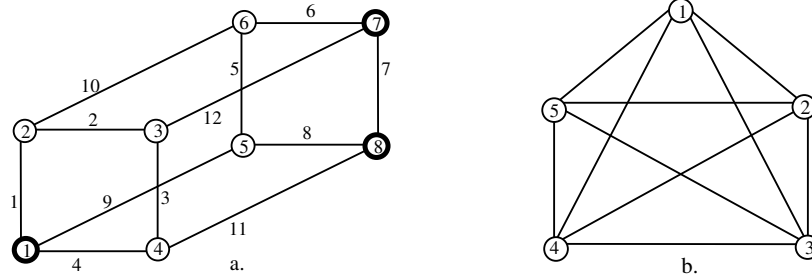


Figure 1: a. H_3 ; b. complete graph with 5 nodes K_5

Here we define three states for the network: *UP* if all terminals are connected to each other; *DOWN1*, if any two terminals are connected, but disconnected from the third, and *DOWN0* if all three terminals become isolated from each other. #

2 D-spectrum and Marginal D-spectra

The definitions below were introduced in [12]. Let $i = 1, 2, \dots, n$ be the numbers of components subject to failure. Denote these components as e_1, e_2, \dots, e_n .

Definition 2.1

Let $\pi = (e_{i_1}, e_{i_2}, \dots, e_{i_n})$ be a permutation of system components. Suppose that initially they all are *up*. Start turning them *down* by moving along π from left to right. Fix the first element e_{i_r} when the system state becomes *DOWN1*. The *ordinal number* r of this element in the permutation is called the *first anchor* of π and denoted by $r_1(\pi)$.

Continue the process of moving along π and turning the components to *down*. Fix the second component e_{i_g} when the system state becomes *DOWN0*. The *ordinal number* g of this component in the π is called the *second anchor* of π and is denoted $r_2(\pi)$. #

Remark 1. Obviously, $r_1(\pi) \leq r_2(\pi)$. When the system is a network and its failing components are edges (not the nodes), then always $r_1(\pi) < r_2(\pi)$ since erasing a single edge can not cause the transition from *UP* to *DOWN0*. #

Consider the set of all $n!$ permutations and assign to each permutation probability $1/n!$. Define the probability of the event $A(i, j) = \{r_1(\pi) = i, r_2(\pi) = j\}$ as

$$f_{i,j} = P(A(i, j)) = \frac{\text{\# of permutations with } r_1(\pi) = i \text{ and } r_2(\pi) = j}{n!}. \quad (1)$$

Definition 2.2.

The two-dimensional discrete density function $\mathbf{d} = \{f_{i,j}\}, i, j = 1, 2, \dots, n, i \leq j$, is called the system two-dimensional destruction spectrum (*D-spectrum*).#

Remark 2. D-spectrum is a purely combinatorial characteristic of the system which is completely determined by its structure function. The probabilities $\{f_{i,j}\}$ are induced by a uniform measure on the set of all $n!$ permutations. D-spectrum is completely separated from any information regarding the real stochastic mechanism which governs system failure appearance. [10] has introduced the notion of D-spectrum in a simplified form, namely for the case when the system has only a *single* failure state. Then instead of two-dimensional density \mathbf{d} we obtain a one-dimensional distribution $\{f_1, f_2, \dots, f_n\}$ of the position of the first anchor.#

Definition 2.3.

The marginal distribution $\mathbf{f} = \{f_1, f_2, \dots, f_n\}$ of the first component of the D-spectrum (*i.e.*, of the first anchor) is called the *first D-spectrum*, and the marginal distribution $\mathbf{g} = \{g_1, g_2, \dots, g_n\}$ of the second component of the D-spectrum (*i.e.*, of the second anchor) is called the *second D-spectrum*.#

Correspondingly, $F(x) = \sum_{i=1}^x f_i$ and $G(y) = \sum_{j=1}^y g_j$ for $x=1, \dots, n$, and $y=1, \dots, n$, are called, respectively, the first and second *cumulative D-spectra*.#

Example 2.

Fig. 1, b presents a network which is a complete graph K_5 with 5 nodes and 10 edges. All nodes are terminals, edges are subject to failures. Three states are defined for this network: *UP* - all terminals are connected, *DOWN1* - there are two isolated subnetworks with terminals (clusters), and *DOWN0* - there are three isolated subnetworks with terminals. For example, if all edges incident to a single node fail, and others are *up*, there will be two clusters.

The total number of edge permutations is $10!$ and we use here Monte Carlo simulation to approximate the two-dimensional D-spectrum. The nonzero $f_{i,j}$ values are the following (see [12]): $f_{4,7} = 0.0047$, $f_{4,8} = 0.0194$, $f_{5,7} = 0.0191$, $f_{5,8} = 0.0751$,

$$f_{6,7} = 0.0596, f_{6,8} = 0.2270, f_{7,8} = 0.5951.$$

From here it is easy to find the marginal D-spectra. Its nonzero components are:

$$f_4 = 0.0241, f_5 = 0.0942, f_6 = 0.2866, f_7 = 0.5951, g_7 = 0.0833, g_8 = 0.9167.$$

Let us conclude this section by explaining the probabilistic meaning of the D-spectrum. Imagine that in reality there is a temporal process of component sequential destruction in random order. This might be a sequential "death" of components for the case of i.i.d. component lives, or a sequence of "attacks" on randomly chosen system components, like local earthquakes, road accidents, *etc.*

Let us concentrate on the two-state situation, when we take into account only the transition from UP to its complement:

$$\overline{UP} = DOWN = DOWN1 \cup DOWN0.$$

Let Y be the number of components needed to be turned from up to $down$ to cause the system get $DOWN$. Then, obviously, $f_r = P(Y = r)$ and

$$F(x) = P(Y \leq x).$$

In words: $F(x)$ is the cumulative distribution function (CDF) of the number of components to be destroyed to cause the system get $DOWN$. So, for the K_5 network of Example 2, $DOWN$ may appear only if the number of destroyed links exceeds 3; if this number is 7, the network will be $DOWN$ with probability 1. After destruction of 6 links, $P(DOWN) = f_4 + f_5 + f_6 = 0.4049$. Suppose 7 links have failed. Then the network can be either in $DOWN1$ or in $DOWN0$ state. Then $P(DOWN1) = 1 - g_7 = 0.9167$.#

Let us conclude this section with an important remark.

"Binary" approach to a three-state system.

Our system considered in Section 1 has three mutually exclusive states: UP , $DOWN1$ and $DOWN0$. It can be looked at as a binary $UP/DOWN$ -system if we unite, for example, $DOWN1$ and $DOWN0$ into a single state

$$DOWN = DOWN1 \cup DOWN0.$$

Another option is to declare UP and $DOWN1$ as a new UP state, and its complement $DOWN0$ as a new $DOWN$ state. We will use both these options depending on the particular situation at hand. Of course, it is easy to generalize this three-state situation to $L > 3$ -state [15,17], but for sake of simplicity we consider $L \leq 3$.

Finally, we stress that so far the probabilistic mechanism of component failures was created only by the artificial destruction process on random permutations. It is completely separated from the "real" random mechanism governing components failures. An important model of such a mechanism will be introduced in the next section.#

3. Samaniego's Signature

Samaniego [1,18,19] considers *binary* coherent systems consisting of n components whose lifetimes are independent identically distributed (i.i.d.) random variables X_1, X_2, \dots, X_n with common CDF $H(t) = P(X_i \leq t)$. Below is his definition of signature [19], p. 21:

Definition 3.1: *Signature.*

Assume that the lifetimes of system's n components are i.i.d. according to the continuous distribution $H(t)$. The signature \mathbf{s} of the system is an n -dimensional probability vector whose i -th element s_i is equal to the probability that the i -th component failure causes the system to fail. In brief, $s_i = P(T = X_{(i:n)})$, where T is the failure time of the system and $X_{(i:n)}$ is the i -th order statistic of the n component failure times.#

The signature was first introduced by F. Samaniego in his paper [18]. The principal fact discovered in this paper is the following surprising formula for the CDF of system lifetime T :

$$P(T \leq t) = \sum_{i=1}^n s_i \cdot P(X_{(i:n)} \leq t). \quad (2)$$

The proof of (2) is easily obtained by the use of the Law of the Total Probability, see e.g., [19], p 26.

An equivalent of signature was later suggested in [8] and termed system ID (Internal Distribution).

Several interesting facts follow from Definition 3.1. and (2). The i.i.d. assumption implies that system components fail in *random* order, and all possible $n!$ such orders are equally probable. [19] says that "... we may obtain s_i as the ratio of n_i , the number of orderings for which the i th component failure causes system failure, to $n!$, the total number of orderings of the n failure times". From here it follows that $\mathbf{s} = \{s_1, s_2, \dots, s_n\}$ is a discrete density function.

Returning to the D-spectra and considering its binary version (i.e., the UP/DOWN situation), where $DOWN = DOWN1 \cup DOWN0$ we conclude immediately that

$$\mathbf{s} = \mathbf{f}, \text{ i.e. } s_i = f_i, i = 1, \dots, n.$$

In other words, the signature coincides with the first marginal D-spectrum. Similarly, if we define system states as $DOWN := DOWN0$ and its complement as $UP := \overline{DOWN}$, then the signature for this case will coincide with the second marginal D-spectrum.

The next surprise contained in (2) comes from some algebraic manipulations with it, see [11], Chapter 8.

Recall the formula for the CDF of the order statistics [7]:

$$P(X_{(r:n)} \leq t) = \sum_{i=r}^n \frac{n!}{i!(n-i)!} [H(t)]^i \cdot [1 - H(t)]^{n-i}. \quad (3)$$

Substitute (3) into (2) and collect all terms having $[H(t)]^i \cdot [1 - H(t)]^{n-i}$ as a multiple. Denote by $S_x = s_1 + s_2 + \dots + s_x$, the so-called *cumulative signature* and $H(t) = q$. Then (2) takes the following simple form:

$$P(T \leq t) = \sum_{x=1}^n S_x \cdot \frac{n!}{x!(n-x)!} q^x (1-q)^{n-x}. \quad (4)$$

Now fix the time parameter t and put $t = t_0$. Then $P(T \leq t_0)$ is the probability that system lifetime does not exceed t_0 , or that at t_0 the system is *DOWN*. (More precisely, the system is *DOWN* at $t_0 + 0$). $q = H(t_0)$ is the probability that a component is *down* at t_0 . We see, therefore, that the probability that the system is *DOWN* at t_0 depends *only* on the cumulative signature and the parameter $q = H(t_0)$.

Suppose now that we have a binary coherent system and carry out independent "lottery" for each of its n components: with probability q the component is declared to be *down* and with probability $p = 1 - q$ - to be *up*. Then the probability $Q(q)$ that the system is *DOWN* equals

$$P(DOWN) = Q(q) = \sum_{x=1}^n S_x \cdot \frac{n!}{x!(n-x)!} q^x (1-q)^{n-x}.$$

If we recall that $s_i = f_i$ and therefore $S_x = f_1 + \dots + f_x = F(x)$, this formula takes the form

$$Q(q) = \sum_{x=1}^n F(x) \cdot \frac{n!}{x!(n-x)!} q^x (1-q)^{n-x}. \quad (5)$$

Now forget for a moment about signatures and D-spectra and consider a binary coherent system in which each component, independently of others, is *down* with probability q . If the system is *DOWN*, it must be in one of its failure sets (cut sets), *i.e.*, system state is $\mathbf{x} : \varphi(\mathbf{x}) = 0$. These failure sets can be classified according to the number of failed components in them. For a failure set \mathbf{x} , this number equals $n - |\mathbf{x}|$. Denote by $A(r)$ the number of failure sets of the system which contain exactly r failed components.

For example, consider a bridge from Example 2. It has two failure sets of size 2, namely (1, 2), (4, 5), both are min cuts. There are eight failure sets of size 3:

(1, 2, 3), (1, 2, 4), (1, 2, 5), (3, 4, 1), (3, 4, 2), (3, 4, 5), (1, 3, 5), (2, 3, 4).

Thus here $A(2) = 2$, $A(3) = 8$. It is easy to see that $A(4) = 5$ and $A(5) = 1$, $A(1) = 0$.

Now system *DOWN* probability can be written as

$$Q(q) = \sum_{r=1}^n A(r) q^r (1-q)^{n-r}. \quad (6)$$

Comparing (5) and (6), we arrive at the following

Theorem 3.1. For $x=1, 2, \dots, n$,

$$F(x) \cdot n! / x!(n-x)! = A(x). \quad (7)$$

We see, therefore, that the cumulative D-spectrum (or equivalently, the cumulative signature) are expressed via the system combinatorial parameter $A(x)$ which can be derived from its structure function. This fact was noticed by many authors, see *e.g.*, [4], [11], chapter 8, [19], p. 80.

Before we demonstrate how this fact can be used in system reliability calculations, let us have another look at (7). Write

$$F(x) = A(x) / (n! / (x!(n-x)!))$$

The denominator is the number of ways to select a set of size x from a set of n components. The numerator is the number of sets with x elements which, if turned *down*, create a failure (cut set) of size x . Imagine that a set of x components is chosen randomly and destroyed by an "attack". Then $F(x)$ is the probability that this attack will destroy the whole system. This interpretation is important in many applications, see *e.g.*, [5], page 435 and [17].

4 Computing System Reliability

4.1 Network Resilience

Suppose we have a network consisting of identical components (edges or nodes) subject to random and independent failures. Slightly modified definition of network *resilience* given in [5], p.435 is the following:

The probabilistic resilience $res_{prob}(\mathbf{N}, \alpha)$ is the largest number of component failures such that \mathbf{N} is still connected with probability $1 - \alpha$.

The first cumulative D-spectrum of the network described in Section 2 ideally suits to finding $res_{prob}(\mathbf{N}, \alpha)$. Take for example the K_5 network (**Example 2**) where the failure is defined as the loss of terminal connectivity. The cumulative D-spectrum equals $F(1) = F(2) = F(3) = 0$, $F(4) = 0.0241$, $F(5) = 0.1183$, $F(6) = 0.4049$, $F(7) = \dots = F(10) = 1$. Suppose $\alpha = 0.1$. Then, obviously,

$$res_{prob}(\mathbf{N}, 0.1) = 4.$$

The so-called *relative resilience* is defined as

$$\eta = \text{res}_{\text{prob}}(\mathbf{N}, \alpha) / n = 4 / 12 = 0.333.$$

In words: K_5 network survives the failure of a third of its edges with probability ≥ 0.9 .

4.2 System Failure Probability as a Function of q

Suppose that system components fail independently with probability q . The most important reliability characteristic of the system is its failure probability $Q = P(\text{DOWN})$ as a function of q . Let us demonstrate how this characteristic is computed using D-spectrum (signature).

Let us consider again the K_5 network (**Example 2**), for which the failure is defined as its disintegration into three isolated subnetworks, *i.e.*, we distinguish two states: DOWN0 and its complement $UP = \overline{\text{DOWN0}}$. All we have to know is system second D-spectrum $G(x)$, $x = 1, \dots, 10$, which is

$$G(1) = \dots = G(6) = 0, G(7) = 0.0833, G(8) = G(9) = G(10) = 1.$$

By (5),

$$P(\text{DOWN0}) = Q(q) = \sum_{x=7}^{10} G(x) q^x (1-q)^{10-x} \cdot 10! / (x!(10-x)!).$$

This formula can be easily tabulated. Here are some results:

$$Q(0.1) = 1.1 \cdot 10^{-6}; Q(0.2) = 0.000143; Q(0.3) = 0.0234.$$

This system is quite reliable. For example, $Q(0.5) = 0.0645$. Even if component failure probability is 0.5, the system is UP with probability close to $1-Q = 0.935$.

4.3 B-P Approximation

Reference [9] describes the following approximation to system reliability. Suppose that component failure probabilities are of the same magnitude, which formally can be written as $q_i = \alpha_i \cdot \varepsilon_i$, $i = 1, \dots, n$. Imagine also that we have a sequence of systems numbered $s = 1, 2, \dots$, of *increasing* reliability, *i.e.*, with decreasing component failure probability. This is formalized as the assumption that $\varepsilon_s \rightarrow 0$ as $s \rightarrow \infty$.

Suppose that the minimal size of system min-cut is r and C_j , $j = 1, \dots, m$ are these minimal cuts. Then as $\varepsilon \rightarrow 0$,

$$R = 1 - \varepsilon^r \sum_{j=1}^m \prod_{s \in C_j} q_s + o(\varepsilon^r). \quad (8)$$

In words: when the system is highly reliable, the main contribution to its failure probability comes from min-size min-cuts, and each min-cut C_j contributes $\prod_{s \in C_j} q_s \cdot \varepsilon^r$. This approximation was suggested long ago by Burtin and Pittel (BP) in [6] in a slightly different form. Quite often, it produces surprisingly accurate reliability estimates for very reliable systems.

Consider, for example, the cubic network of **Example 1**. Define its failure as the loss of terminal connectivity. The nonzero terms of the first D-spectrum are:

$$f_3 = 0.0141, f_4 = 0.0623, f_5 = 0.1847, f_6 = 0.3442, f_7 = 0.2352, f_8 = 0.1114,$$

$$f_9 = 0.0388, f_{10} = 0.0093.$$

Let q be the edge failure probability. The first nonzero term of the cumulative spectrum is $F(3) = f_3 = 0.0141$. Formula (7) gives us the number of minimal size failure sets:

$$A(3) = F(3)12!/(3!9!) = 3.1.$$

Rounding to the nearest integer, we get $A(3)=3$, (check with Fig. 1 a.). Thus, we take the approximation to R as $R^* = 1 - 3q^3$. Let us compare it with the approximation result

$R_0 = 1 - Q$ obtained using (6). For $q = 0.02$, $R_0 = 0.999974$, $R^* = 0.999976$;

For $q = 0.05$, $R_0 = 0.999547$, $R^* = 0.999625$; For $q = 0.1$, $R_0 = 0.995842$, $R^* = 0.997$.

So, for $q \leq 0.1$, the approximation is satisfactory.

4.4 Bounds on Q for Imprecise Data

Suppose, it is known that component *down* probability q is not known exactly (as it usually takes place in practice) and lies in the interval in $q \in [q_{\min}, q_{\max}]$. Since system reliability is a monotone function of its component reliability [2], we have the following bounds on $P(DOWN) = Q(q)$ (see [11]):

$$Q(q_{\min}) \leq P(DOWN) = Q(q) \leq Q(q_{\max}), \quad (9)$$

where $Q(q)$ is given by (6).

Example 3: K_6 network.

Consider a network described by a complete graph K_6 . It has 6 nodes, 15 edges, all nodes are terminals, edges are subject to failure. *DOWN* is defined as loss of terminal connectivity. [11], p. 190 gives the data on its the D-spectrum, from which follows that $F(4) = 0$, $F(5) = 0.00205$, $F(6) = 0.012205$, $F(7) = 0.042025$, $F(8) = 0.113962$, $F(9) = 0.269174$, $F(10) = 0.568008$, $F(11) = \dots = F(15) = 1$.

By (6),(7)

$$Q(q) = 15! \sum_{x=5}^{15} F(x) q^x (1-q)^{15-x} / (x!(15-x)!).$$

Suppose that $q_{\min} = 0.95 \cdot q$ and $q_{\max} = 1.05 \cdot q$ which establishes a 5% "tolerance" interval on q . Fig. 2 below presents three curves for $Q(q_{\min})$, $P(DOWN) = Q(q)$ and $Q(q_{\max})$.

We see that for a rather large interval $q \in [0, 0.45]$ these bounds are not far from each other. For example, for $q = 0.4 \pm 0.02$, the bounds are $[0.05, 0.08]$, which may be quite acceptable for practical purposes.#

5 Birnbaum Importance Measure and its Combinatorial Representation

In this section we extend our combinatorial approach to compute the component Birnbaum Importance Measure (BIM) [2, 3, 11, 16].

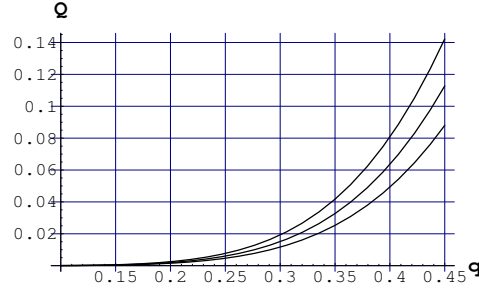


Figure 2: Bounds on the *DOWN* probability of K_6

Suppose a coherent system consists of n independent components, and component i has reliability $p_i, i = 1, \dots, n$. Let the system reliability be

$$R = \Psi(p_1, p_2, \dots, p_n).$$

Then component j BIM has been originally defined [3] as

$$BIM_j = \frac{\partial \Psi(p_1, p_2, \dots, p_n)}{\partial p_j} \quad (10)$$

The importance of BIM follows from its very definition because the knowledge of the

derivative is crucial for solving optimal system design problems, see for example [11]. However, the use of BIM is strongly limited by the necessity to know the analytic form of $\Psi(\mathbf{p})$. This difficulty is eliminated if $p_i \equiv p$ by means of introducing so-called component BIM-spectra which are closely related to the D-spectra [11,14,16].

Let us skip some technical details given in [11,14,16] and note that

$$A(x) = A(x; 0_j) + A(x; 1_j),$$

where $A(x; 0_j)$ is the number of failure sets (cut sets) which have exactly x down components and component j is among the down components, and $A(x; 1_j)$ is the number of failure sets (cut sets) which have exactly x down components and component j is **not** among the down components.

Now we can express the *DOWN* probability (6) as

$$Q(q) = q \cdot \sum_{x=1}^n A(x; 0_j) q^{x-1} (1-q)^{n-x} + (1-q) \sum_{x=1}^n A(x; 1_j) q^x (1-q)^{n-x-1}. \quad (11)$$

In the second sum, the summation goes in fact up to $x = n-1$ since $A(n; 1_j) = 0$. Let us now connect $A(x; 0_j)$ and $A(x; 1_j)$ with the cumulative D-spectrum. Recalling the definition of random variable Y in Section 2, we see that the probability that system is *DOWN* when x components are down splits into two probabilities

$$F(x; 0_j) = P((Y \leq x) \cap (\text{component } j \text{ is down})), \quad (12)$$

$$\text{and} \quad F(x; 1_j) = F(x) - F(x; 0_j) = P((Y \leq x) \cap (\text{component } j \text{ is up})). \quad (13)$$

$$\text{Therefore,} \quad A(x; 0_j) = F(x; 0_j) \frac{n!}{x!(n-x)!}, \quad A(x; 1_j) = F(x; 1_j) \frac{n!}{x!(n-x)!}. \quad (14)$$

After some calculations, we arrive at the following expression for BIM_j [14]:

Theorem 5.1

$$BIM_j = n! \left[\sum_{x=1}^n F(x; 0_j) q^{x-1} (1-q)^{n-x} / (x!(n-x)!) - \sum_{x=1}^n (F(x) - F(x; 1_j)) q^x (1-q)^{n-x-1} / (x!(n-x-1)!) \right]. \# \quad (15)$$

Definition 5.1 [14]: The collection $F(x; 0_j)$, $x = 1, 2, \dots, n$, is called the *BIM-spectrum* of component j .#

We note that the BIM of a component depends on its BIM-spectrum and on the q value. Comparing (15) for i and $j \neq i$, we arrive at the following

Corollary 5.2.

If $F(x; 0_j) \geq F(x; 0_i)$ for all $x = 1, 2, \dots, n$,

then for all $q \in (0, 1)$, $BIM_i \geq BIM_j$.# (16)

In other words: if the BIM-spectrum of component i dominates the BIM-spectrum of component j , i is more important than j , no matter what the values of q are.

Example 1-continued: H_3 network.

Let us combine the *DOWN1* and *DOWN0* into a single *DOWN* state. Network failure is defined, therefore as the loss of terminal connectivity. The *UP* state is the situation when all three terminals are connected to each other.

Table 1: Simulated BIM-spectra for edges 7, 4, 12. M=10,000 Replications

x	$F(x; 0_7)$	$F(x; 0_4)$	$F(x; 0_{12})$	x	$F(x; 0_7)$	$F(x; 0_4)$	$F(x; 0_{12})$
3	0.0099	0.0052	0.0048	8	0.6639	0.6407	0.6278
4	0.0489	0.0310	0.0282	9	0.7514	0.7441	0.7405
5	0.1639	0.1201	0.1140	10	0.8348	0.8320	0.8329
6	0.3899	0.3150	0.2952	11	0.9169	0.9158	0.9160
7	0.5543	0.5009	0.4798	12	1	1	1

Table 1. presents the estimated BIM-spectra for three edges 7, 4, and 12, based on M = 10, 000 Monte Carlo replications. There is a clear domination of BIM_7 over BIM_4 and the latter over BIM_{12} . Therefore, edge 7 is the "most important" (which is obvious from Fig. 1,a.), edge 4 is the "second important" and dominates edge 12:

$$7 \succ 4 \succ 12$$

Analyzing the BIM-spectra of all 12 edges of the network, we arrive at the conclusion that there are four groups of edges ranked by their importance. The first group consists of a single edge 7 connecting two adjacent terminals. In the second group, there are four equally important edges 4, 11, 9, 8; all they are incident to terminals 8 or 1. The third group consists of edges 1, 6, 12.

$$\{7\} \succ \{4, 11, 9, 8\} \succ \{1, 6, 12\}.$$

The remaining edges (2,3,5,10) are not incident to any terminal, and they are the least important. If, for example two components are to be reinforced, the best choice would be edge 12 and any edge from the second group.#

6 Monte Carlo for D-spectra and BIM-spectra

Exact computation of system signatures, D-spectra and BIM-spectra is an NP-hard problem. [19], p. 87, presents formulas for network reliability calculation, for two networks with 9 nodes and 27 edges. They are based on enumeration of all network minimal path sets (which is also NP-hard) and present the results in a form of polynomials having alternating signs and large coefficients. There are papers tabulating the signatures for coherent system, see e.g., [13], but this works good only for rather small coherent systems. We believe that the only practical way to calculate the signatures and spectra is approximating them using Monte Carlo (MC) methodology. The reader is referred to the book [11] which contains a series of MC algorithms and examples of spectra calculation.

All MC algorithms for approximating D-spectra are in fact quite straightforward. To estimate $F(x)$, simulate M random permutations $\pi = (i_1, i_2, \dots, i_n)$ of component numbers and imitate a sequential destruction of components by moving along a permutation from left to right and by remembering the number N_i of such permutations that the system went *DOWN* on the i -th step of the destruction process. Afterwards, as an MC estimate of $F(x)$ take the ratio

$$\hat{F}(x) = (N_1 + \dots + N_x) / M.$$

To approximate BIM-spectra, modify the above procedure and count the number of permutations $M(x; 0_j)$ equal to the number of permutations such that the system went *DOWN* during the first x failures and component j was among these x components.

The algorithmic "trick" that made it possible to accelerate the destruction process [11] is based on a special property of the optimal spanning tree for network-type systems allowing to find out the permutation anchor by constructing only a single tree for each simulated π . The number of replications M needed to obtain accurate approximations was between 10^5 and 10^8 , depending on the system size. Since the signature (D-spectrum) is an inherent system parameter, spending more computer time for the MC approximation is never a critical issue.

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Ilya B. Gertsbakh is currently Professor Emeritus in the Department of Mathematics at Ben Gurion University of the Negev, Israel. He is author or coauthor of six books, one of which is *Models of Network Reliability: Analysis, Combinatorics, and Monte Carlo* (coauthored with Yoseph Shpungin), published by CRC Press, in 2009. Ilya Gertsbakh has published some 70 papers on topics in operations research, reliability, applied probability and applied statistics. He received M.Sc degrees in mechanical engineering and mathematics (1961) from Latvian State University in Riga, Latvia, and the Ph.D. degree (1964) in applied probability and statistics from Latvian Academy of Sciences, also in Riga.

Yoseph Shpungin is currently Head of Department of Software Engineering at Sami Shamoon College of Engineering in Beer-Sheva, Israel. He is a coauthor of two books, the recent of which is "Network Reliability and Resilience" published in the series Springer, *Briefs in Electrical and Computer Engineering*, September 2011, Springer Heidelberg Berlin. Yoseph Shpungin has published some 20 papers on topics of Monte Carlo, network reliability and operations research. He received his Ph.D. in Mathematics from Ben Gurion University in 1997