

A Theoretically Appropriate Poisson Process Monitor

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Abstract: Because the probability of Type I error is not evenly distributed beyond upper and lower three-sigma limits the c chart is theoretically inappropriate for a monitor of Poisson distributed phenomena. Furthermore the normal approximation to the Poisson is of little use when c is small. These practical and theoretical concerns should motivate the computation of true error rates associated with individuals control assuming the Poisson distribution.

Keywords: *Attributes, control charts, economic design, Poisson*

1. Introduction

The probability of Type I error is not evenly distributed beyond upper and lower statistical quality control limits of traditional charts for nonconformities, given Poisson distributed defects. Also the normal approximation to the Poisson is of little use when the expected number of nonconformities is small. Therefore we are motivated to design low-cost, theoretically appropriate control charts that assume the Poisson distribution. Such economic design requires computation of true error rates and assumptions about the relative costs of errors Type I and II.

Suppose that nonconformities or defects occur in an inspection unit according to the Poisson distribution: $p(x) = e^{-c} c^x / x!$, $x = 0, 1, 2, \dots$, where x is the number of defects, and $c > 0$ defines the Poisson distribution (mean and variance). Assuming a standard value for c , the traditional c chart for nonconformities is defined as follows: Upper control limit = $c + 3 \text{ SQRT}(c)$, Centerline = c , and lower control limit = $c - 3 \text{ SQRT}(c)$.

Because the c chart effectively assumes the normal distribution for a counting process calculations can yield a negative value for the lower control limit (LCL) in which case it is suggested that we set $\text{LCL} = 0$. For example this is one practical consequence of an ill advised normal approximation. It should motivate economic design of theoretically appropriate quality control for Poisson distributed defects.

2. Relevant Literature

The relevant design literature can be divided among three areas: statistical quality control charts, economic quality control charts, and economic-statistical quality control charts. Kaminsky, *et al.* noted that in some instances using a shifted geometric distribution may be more appropriate for Poisson distributed defects, because traditional c charts tend to underestimate process variability [1]. Results from their study showed that compared to more traditional charts false alarm rates were reduced by assuming the geometric distribution. Later a method which dealt explicitly with the number of observations between defects was introduced by Nelson and found to be particularly good for the case of near-zero defects [2]. Chang and Gan further extended these ideas by proposing a scheme for the cumulative count [3], and with every technological advance charts became more cumbersome and perhaps difficult to justify as departures from the c chart.

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Straightforward moving averages have been used to monitor nonconformities and compared to the c chart [4], and similar improvements were found when exponentially weighted moving average (EWMA) control charts were designed and analyzed [5]. Woodall provides an extensive literature review of control charts designed for observations including the EWMA and cumulative sums (CUSUM) [6]. More recently authors have focused on enhancements of the original c chart [7].

Developments in the economic design of quality control charts seek to reduce the cost of process control. Traditionally the four main components of cost are sampling, the false alarm, finding and correcting an assignable cause, and the cost of a defective item. These components are used to determine an economic combination of sample size, control limits, and inter-sample interval. Authors have compared economic designs for CUSUM and geometric moving averages to find that \bar{X} is better to detect large shifts [8]. However many economic models can be prohibitively intricate. Taken separately each of the four cost components can be difficult to estimate accurately. For this reason we favor a simplified approach to monitoring individual observations of a counting process, where the only costs to consider are those associated with errors Type I and II.

The trouble with economic quality control has been that minimum cost solutions can actually run counter to business constraints. For example Williams, *et al.* displayed an optimal solution to produce 64% defectives [9]. The design might have been optimal, but the results would not have conformed to the company's objectives with respect to customer satisfaction. According to Ho and Case in economic-statistical design the loss function of a process is minimized subject to three main constraints: minimum power, maximum Type I error rate, and average time to detect a shift [10]. An excellent example of this proposed an optimization model for the joint design of \bar{X} and R charts [11]. In the words of the author, "The actual users of control charts are interested in designs that are simple to understand and use." We have found this ultimate goal to be entirely compatible with theoretically appropriate methods for economic quality control of a Poisson process.

Demerit control limits for Poisson-distributed defects have already been presented and discussed in the context of economic design [12, 13]. The work described here is also related to demerit systems assuming the binomial distribution that were recently introduced and applied to medication error severity data [14, 15]. The particular emphasis we place on economic design might have been most recently featured in a diversity monitor with known errors for process variability observed in categorical data [16].

3. Methods and Results

We examined the concept of a theoretically appropriate monitor for the Poisson process by first arbitrarily choosing some values for c , and computing the associated Type I error rates for combinations of reasonable upper control limits (UCL) and lower control limits (LCL). An observation greater than the UCL or less than the LCL is considered to be out of control. Under the assumption that no real shift has occurred an out of control signal is a Type I error. Assuming a value for c we can find the probabilities associated with observing any number of defects and so the Type I error rate. See Table 1 for an example when $c = 2$.

Next for every combination of shift from c to c_I we computed the probabilities of Type II errors. An observation between or equal to control limits is considered to be in control. Under the assumption that a shift has actually occurred an in control signal is a Type II error. It is convenient that the Type II error associated with a shift from c to c_I complements the Type I error associated with $c = c_I$. For example the Type II error rates

associated with a shift from any c to $c_I = 2$ are equal to “one minus” the values in Table I.

Assuming equal cost errors we summed the error rates Type I and II for every combination of control limits and shift to discover the minimum cost and associated control limits. See Table 2 for an example of the shift from $c = 2$ to $c_I = 6$. Obviously the economic design among those in Table 2 has the minimum cost of 0.2941: LCL is None, and UCL is 4. Consider for example the comparable c chart with LCL approximately -2 (practically none) and UCL approximately 6. We know from Table 2 cost associated with the c chart is 0.4622 for this example, approximately 57% more expensive than the Poisson process monitor presented here. Table 3 shows what are the minimum cost control limits for combinations of shifts from c to c_I .

Table 1: Example Type I Error Rates (when $c = 2$)

	UCL = 1	2	3	4	5	6	7
LCL = 5							0.9880
4						0.9639	0.9519
3					0.9098	0.8737	0.8616
2				0.8196	0.7293	0.6932	0.6812
1			0.7293	0.5489	0.4586	0.4226	0.4105
0		0.7293	0.4586	0.2782	0.1880	0.1519	0.1399
None	0.8647	0.5940	0.3233	0.1429	0.0526	0.0166	0.0045

Table 2: Total Costs Assuming Equal Cost Errors (shift from $c_0 = 2$ to $c_I = 6$)

	UCL = 1	2	3	4	5	6	7
LCL = 5							1.1486
4						1.1245	1.2731
3					1.0436	1.1682	1.3168
2				0.9088	0.9524	1.0770	1.2255
1			0.7739	0.6827	0.7264	0.8509	0.9995
0		0.7442	0.5181	0.4269	0.4706	0.5951	0.7437
None	0.8647	0.6113	0.3853	0.2941	0.3377	0.4622	0.6108

4. Conclusions and Future Work

We have presented the concept of an unconstrained, economically designed, theoretically appropriate monitor for the Poisson process. Techniques focusing on the Poisson process have been well studied, but the foundation of these studies seems to have been shared with the c chart, a theoretically inappropriate normal approximation. What we have presented here is an alternative which is theoretically appropriate in that its assumptions do not deviate from the true distribution of what is being monitored. Future work should include additional values for c ; upward and downward shifts to and from each parameter would be evaluated. Another idea is to look for a good meta model of results like the ones appearing in Table 3. For example it would be useful to know if the variation in minimum cost can be understood as a smooth function of c and c_I . Such a function might make the enumeration described in Methods and Results unnecessary. One might also like to know if results change in a simple way according to different error costs. Finally the work here would be strengthened by showing an application where

interesting data conform well to Poisson distributions like the ones we consider, and theoretically appropriate monitoring decisions can be made more intelligently, according to economic design.

Table 3: Minimum Costs (assuming equal cost errors)

c_0	Type I error rate	c_1	LCL	UCL	Type II error rate	Minimum cost
0.5	0.3935	1	None	1	0.3679	0.7613
0.5	0.0902	2	None	2	0.4060	0.4962
0.5	0.0143	6	None	3	0.0620	0.0764
1	0.2642	2	None	2	0.4060	0.6702
1	0.0803	6	None	3	0.0620	0.1423
2	0.1429	6	None	4	0.1512	0.2941
6	0.1512	2	3	None	0.1429	0.2941
6	0.0620	1	2	None	0.0803	0.1423
6	0.0620	0.5	2	None	0.0143	0.0764
2	0.4060	1	1	None	0.2642	0.6702
2	0.4060	0.5	1	None	0.0902	0.4962
1	0.3679	0.5	0	None	0.3935	0.7613

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