Hierarchical Bayesian Reliability Analysis of Binomial Distribution based on Zero-Failure Data

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Abstract

The aim of this paper is to develop a new hierarchical Bayesian estimation method under symmetric entropy loss function for reliability of the binomial distribution. With the rapid development of manufacturing techniques, some electric products are highly reliable, and thus zero-failure data often occur when putting them in censored lifetime tests. Based on zero-failure data, the reliability analysis is very important for manufacturing. The hierarchical Bayesian estimator is regarded as a robust estimating method, but many existing robust Bayes estimators are complex and difficult to be utilized in practice. The contribution of this article is to present an easy hierarchical Bayesian estimator for reliability of the binomial distribution when reliability has a negative log-gamma prior distribution. Finally, a practical example is provided to show the feasibility and robustness of different estimators.

Keywords: binomial distribution; bayes estimation; symmetric entropy loss function; hierarchical bayesian estimation

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1. Introduction

Reliability tests often receive all kinds of truncated data. In the time truncation test, zero-failure data is often encountered, especially in high reliability and small sample problems. Martz and Waller [1] first put forward the Bayesian method to deal with the statistical analysis of the model with zero-failure data. In recent years, it has attracted more and more attention as manufacturing techniques rapidly increase [2-5]. The Bayesian statistics method has been widely applied in various fields, such as computer science [6], reliability engineering [7-8], safety engineering [9], medical science [10], and finance [11].

The hierarchical prior distribution was first proposed by Lindley and Smith [12], and they not only developed the hierarchical Bayes estimation method, but also proved that it performs better than the ordinary Bayes estimation. Han [13] introduced an idea for constructing the hierarchical prior distribution of reliability with zero-failure data. Since then, hierarchical Bayesian statistical inference methods have received much attention from scholars [14-16]. However, the hierarchical Bayesian method is complex because of integration, which is usually difficult to compute. Though Markov Chain Monte Carlo (MCMC) and other simulation methods can help solve the above-mentioned complex integral [17-18], it is still not easy to be implemented in many cases. To overcome this drawback, Han [19] proposed the E-Bayes method and proved that it is a feasible method to substitute the hierarchical Bayesian method. However, when using the E-Bayes method for any distribution, we still need to prove the rationality of the estimator [20], which is also a difficult work. If we can derive a simple expression for the hierarchical Bayesian estimation in a straightforward manner, we can solve these problems. The main purpose of this paper is to present a simple method for estimating the reliability parameter of the binomial distribution using the hierarchical Bayesian method.

In some cases, it is not easy to determine products’ life types. Sometimes, although the life distribution type of the product is known, it is only the number of failures, and there is no exact time of failure. At this time, we can obtain the estimation of reliability by means of the non-parametric method. Let $X$ be the number of failures of $n$ independent trials,
and it is distributed with the binomial distribution, as follows:

\[ P(X = x) = \binom{n}{x} R^x (1-R)^{n-x}, \ x = 0,1,2,\ldots,n \]  

(1)

In reliability analyses, \( R \) is often called the reliability.

The main task of this article is to study the hierarchical Bayesian reliability analysis for the binomial distribution using a new proposed prior distribution.

The rest of the article is organized as follows: Section 2 recalls the concept and property of the symmetric entropy loss function and then introduces the negative log-gamma prior distribution. This section also proposes the hierarchical prior distribution by using the increasing function method based on the negative log-gamma distribution. Section 3 studies the hierarchical Bayesian estimation of the reliability of the binomial distribution. Section 4 gives Monte Carlo simulations and an applied example to observe the performance of the proposed estimators. Finally, concluding remarks are made in Section 5.

2. Preliminary Knowledge

2.1. Loss Function

Loss function is one of the most important elements of Bayesian statistical inference. Squared error loss (SEL), LINEX loss, and entropy loss functions are the most often used loss functions in Bayesian estimation [21-23]. This paper will study the Bayesian reliability analysis of the binomial distribution under a symmetric entropy loss function, which has the following form [24-25]:

\[ L(R, \delta) = \frac{\delta}{R} + \frac{R}{\delta} - 2 \]  

(2)

Where \( \delta \) is an estimator of unknown parameter \( R \).

Lemma 1 [24] Let \( \delta \) be an estimator of the parameter \( R \), \( \pi(R) \) is a prior distribution of parameter \( R \), then under the symmetric entropy loss (2), the Bayesian estimator of \( R \) is

\[ \hat{R}_B = \left[ \frac{E(R|X)}{E(R^{-1}|X)} \right]^{0.5} \]  

(3)

The Bayes estimator is unique, when assuming the Bayes risk \( r(\delta) < +\infty \).

2.2. Prior Distribution

Prior distribution is a key element in a Bayesian analysis. The Bayesian approach often assumes that we can obtain some prior knowledge about the unknown parameter by investigating past experiences, statistical simulation experiments, or experts’ knowledge. Prior distribution is often used to display this prior knowledge.

For the model containing zero-failure data, Han [13] proposed the increasing function method to construct the prior distribution of reliability. The theoretical basis of the increasing function method is to choose the increasing function as the kernel of the prior density of \( R \). It conforms with the high possibility of larger values of \( R \) under zero-failure or very few failure data and the low possibility of smaller values of \( R \). Therefore, the choice of a more reasonable prior distribution is the focus of our attention. Because the negative logarithm gamma distribution is a good distribution, it can fit all kinds of density functions on (0, 1). This paper utilizes it to model the prior distribution of the reliability \( R \). That is, \( R \) has the negative logarithm gamma prior distribution, and its probability density function is
\[ \pi(R; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} R^{\beta-1}(-\ln R)^{\alpha-1}, \quad 0 < R \leq 1 \] (4)

Figure 1 illustrates the shape of the negative logarithm gamma density function. We can see that it has various shapes when we set different values for \( \alpha \) and \( \beta \).

According to Han [13], the values of hyperparameters \( \alpha \) and \( \beta \) should be selected such that \( \pi(R; \alpha, \beta) \) is guaranteed to be an increasing function of \( R \).

The derivative of \( \pi(R; \alpha, \beta) \) over parameter \( R \) is

\[ \frac{d\pi(R; a, b)}{dR} = \frac{\beta^\alpha}{\Gamma(\alpha)} R^{\beta-2}(-\ln R)^{\alpha-2}[\beta - 1)(-\ln R) + (1 - \alpha)] \] (5)

Since \( 0 < R \leq 1 \), then \( 0 < \alpha \leq 1 \) and \( \beta > 1 \) will result in \( \frac{d\pi(R; a, b)}{dR} > 0 \). That is, \( \pi(R; \alpha, \beta) \) is an increasing function of \( R \) when \( 0 < \alpha \leq 1 \) and \( \beta > 1 \).

Berger [26] pointed out that the narrower tailed probability density function as a prior distribution often leads to worse robustness. Considering the robustness of the Bayes estimator, the value of the hyperparameter \( \beta \) should not be too large. Then, we choose \( \beta \) below some positive constant \( k \). Therefore, we can consider the scope of hyperparameter \( \beta \) in an interval \([0, k]\) (\( k = 2 \) for instance).

3. Bayes Reliability Analysis of Reliability

3.1. Bayes Estimation of Reliability

**Theorem 1** Suppose that the reliability \( R \) of the binomial distribution (1) has the negative logarithm gamma prior distribution (3), with \( 0 < \alpha \leq 1 \) and \( \beta > 1 \). Then, for zero-failure data under the symmetric entropy loss function (2), the Bayes estimator of parameter \( R \) is

\[ \hat{R}_t(\alpha, \beta) = \left( \frac{n + \beta - 1}{n + \beta + 1} \right)^\alpha \] (6)

**Proof** For the case of zero-failure data, the likelihood function of reliability \( R \) corresponding to the binomial distribution (1) is given as [4]:
Hierarchical Bayesian Reliability Analysis of Binomial Distribution based on Zero-Failure Data

According to Equation (7) and Bayes’ Theorem, we can derive the posterior probability density function of parameter $R$ as follows:

$$L(0|R) = R^\alpha$$  \hspace{1cm} (7)

Then

$$h(R|0) = \frac{L(0|R) \cdot \pi(R)}{\int_0^1 L(0|R) \cdot \pi(R) dR}$$

$$= \frac{R^\alpha \cdot \beta^\alpha \cdot R^{\alpha-1} (-\ln R)^{\alpha-1}}{\Gamma(\alpha) \int_0^1 R^\alpha \cdot \beta^\alpha \cdot R^{\alpha-1} (-\ln R)^{\alpha-1} dR}$$

$$= \frac{R^{\alpha+\beta-1} (-\ln R)^{\alpha-1}}{\Gamma(\alpha)}$$

$$= \frac{(n+\beta)^\alpha}{\Gamma(\alpha)} R^{\alpha+\beta-1} (-\ln R)^{\alpha-1}$$

Then

$$E(R|0) = \int_0^1 R \cdot h(R|0) dR = \int_0^1 R \cdot \frac{(n+\beta)^\alpha}{\Gamma(\alpha)} R^{\alpha+\beta-1} (-\ln R)^{\alpha-1} dR = \left( \frac{n+\beta}{n+\beta+1} \right)^\alpha$$

$$E(R^{-1}|0) = \int_0^1 R^{-1} \cdot h(R|0) dR = \int_0^1 R^{-1} \cdot \frac{(n+\beta)^\alpha}{\Gamma(\alpha)} R^{\alpha+\beta-1} (-\ln R)^{\alpha-1} dR = \left( \frac{n+\beta}{n+\beta-1} \right)^\alpha$$

Thus, according to Equation (3), the Bayes estimator of parameter $R$ can be obtained as follows:

$$\hat{R}_h(\alpha, \beta) = \left[ \frac{E(R|X)}{E(R^{-1}|X)} \right]^{1/2} = \left( \frac{n+\beta-1}{n+\beta+1} \right)^{1/2}$$

This completes the proof.

3.2. Hierarchical Bayes Estimation of Reliability

If $R$ has the negative logarithm gamma prior distribution given by (3), how can the values of hyper-parameters $\alpha$ and $\beta$ be determined? Hierarchical prior distribution, first introduced by Lindley and Smith [12], can help solve this problem. It is suggested that a prior distribution may contain hyper-parameter(s). In other words, the hyper-parameters can have its own prior distribution. According to prior discussions, $\alpha$ and $\beta$ satisfy $0 < \alpha \leq 1$ and $\beta > 1$. Furthermore, we assume that $\alpha$ and $\beta$ are independent, $\alpha$ is uniform on $(0,1)$, and $\beta$ is uniform on $(1,k)$. That is, $\pi(\alpha) = U(0,1), \pi(\beta) = U(1,k)$, and $k$ is a given positive constant.

**Theorem 2** For the binomial distribution (1), we suppose that the reliability $R$ has the negative logarithm gamma prior distribution given by (3), with $\pi(\alpha) = U(0,1), \pi(\beta) = U(1,k)$, and $k$ is a positive constant. Then, for zero-failure data, under the symmetric entropy loss function (2), the Bayes estimator of parameter $R$ is

$$\hat{R}_{Hb} = \left[ \frac{G(n+1,k)}{G(n-1,k)} \right]^{1/2}$$

(12)
Theorem According to the assumption, the hierarchical prior probability density function of $R$ can be derived as follows:

$$
\pi(R) = \int_0^1 \pi(R; \alpha, \beta) \cdot \pi(\alpha) \cdot \pi(\beta) \, d\alpha \, d\beta
$$

$$
= \frac{1}{k-1} \int_0^1 \int_0^1 \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{R^{\alpha-1}(-\ln R)^{\beta-1}}{\beta^{\beta-1}} \, d\alpha \, d\beta
$$

(13)

According to Bayes’ Theorem and combining Equation (1) with Equation (13), the posterior probability density function of parameter $R$ can be derived as

$$
H(R | 0) = \frac{1}{k-1} \int_0^1 \int_0^1 \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{R^{\alpha-1}(-\ln R)^{\beta-1}}{\beta^{\beta-1}} \, d\alpha \, d\beta
$$

$$
= \frac{1}{k-1} \int_0^1 \int_0^1 \frac{\beta^\alpha}{\Gamma(\alpha)} \left[ \int_0 R^{\alpha-1}(-\ln R)^{\beta-1} \, dR \right] \, d\alpha \, d\beta
$$

(14)

That is,

$$
H(R | 0) = \frac{1}{k-1} \int_0^1 \int_0^1 \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{R^{\alpha-1}(-\ln R)^{\beta-1}}{\beta^{\beta-1}} \, d\alpha \, d\beta
$$

$$
= \frac{1}{k-1} \int_0^1 \int_0^1 \frac{\beta^\alpha}{\Gamma(\alpha)} \left[ \int_0 R^{\alpha-1}(-\ln R)^{\beta-1} \, dR \right] \, d\alpha \, d\beta
$$

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(15)

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$$

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$$
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$$
= \frac{1}{k-1} \int_0^1 \int_0^1 \frac{\beta^\alpha}{\Gamma(\alpha)} \left[ \int_0 R^{\alpha-1}(-\ln R)^{\beta-1} \, dR \right] \, d\alpha \, d\beta
$$

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H(R | 0) = \frac{1}{k-1} \int_0^1 \int_0^1 \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{R^{\alpha-1}(-\ln R)^{\beta-1}}{\beta^{\beta-1}} \, d\alpha \, d\beta
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$$

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$$
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$$

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$$

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$$

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$$

That is,

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$$

That is,

$$
H(R | 0) = \frac{1}{k-1} \int_0^1 \int_0^1 \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{R^{\alpha-1}(-\ln R)^{\beta-1}}{\beta^{\beta-1}} \, d\alpha \, d\beta
$$

That is,)
Hierarchical Bayesian Reliability Analysis of Binomial Distribution based on Zero-Failure Data

\[
E[R \mid 0] = \int_0^1 R \left[ \int_0^1 \beta^\alpha R^{1+\beta-1}(-\ln R)^{\alpha-1} d\alpha \right] d\beta = \frac{G(n+1,k)}{G(n,k)}
\]

\[
E[R^{-1} \mid 0] = \int_0^1 R^{-1} \left[ \int_0^1 \beta^\alpha R^{1+\beta-1}(-\ln R)^{\alpha-1} d\alpha \right] d\beta = \frac{G(n-1,k)}{G(n,k)}
\]

Therefore, according to Equation (3), the Bayes estimator of parameter \( \hat{R} \) is

\[
\hat{R}_{\text{HB}} = \left[ \frac{E(R \mid X)}{E(R^{-1} \mid X)} \right]^{-1/2} = \left[ \frac{G(n+1,k)}{G(n-1,k)} \right]^{1/2}
\]

This completes the proof.

4. Practical Example

An example drawn from [27] is used to show the performance and robustness of our result. The test was carried out in three batches of a fuze and the number of samples in the three batches are \( n = 15, 27, 100 \), respectively. There are no failure samples. Table 1 lists the estimating results of reliability for the binomial distribution.

<table>
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<th>( k )</th>
<th>( \hat{R}_{\text{HB}} (n=15) )</th>
<th>( \hat{R}_{\text{HB}} (n=27) )</th>
<th>( \hat{R}_{\text{HB}} (n=100) )</th>
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<td>0.000692</td>
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5. Conclusions

The study of reliability analysis for the model containing zero-failure data is an important topic in reliability engineering. Under the assumption of the negative logarithm gamma prior distribution and symmetric entropy loss function, this study derived a Bayesian estimator and hierarchical Bayesian estimator of an unknown reliability parameter for the binomial distribution with zero-failure data. The hierarchical Bayesian estimator obtained in this paper is easy to compute. The practical example shows that the hierarchical Bayesian estimator is robust. In the future, we will study the hierarchical Bayesian estimation of other lifetime distributions, such as the generalized exponential distribution and Topp-Leone distribution, when the testing data is zero-failure data.

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