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An Efficient Moving Optimal Radial Sampling Method for Reliability-Based Design Optimization

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Abstract

Reliability-based design optimization (RBDO) plays an essential role in structure and system design. However, its application in practical engineering is hindered by the huge computational cost in performance function evaluation. In this paper, a moving optimal radial sampling (MORS) method is proposed for RBDO problems to substantially improve the computational efficiency of Monte Carlo simulation (MCS). In MORS, the failure probability and its gradient are calculated using radius based importance sampling (RBIS) method. The initial radius is selected according to the target reliability, which can also be used to check the feasibility of probabilistic constraints afterwards. The arc search scheme in enhanced performance measure approach (PMA+) and linear interpolation scheme are used to calculate the optimal radius of RBIS. After the failure probability and its gradient are calculated, the optimal design is obtained using sequential approximation programming (SAP). The computational capability of the proposed MORS method is demonstrated using a honeycomb crashworthiness design application, a nonlinear mathematical problem and a speed reducer design. The comparison results show that the proposed MORS-SAP method is very efficient and accurate.

Keywords: reliability-based design; optimization; monte carlo simulation; moving optimal radial sampling

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1. Introduction

Deterministic optimization methods have been successfully applied in complex product design problem while satisfying some certain performance constraints in manufacturing industry. However, without quantitatively considering the uncertainties in material properties, geometry size, loadings and boundary condition [29], the optimal design may be either risky while the design has a low probability of constraint satisfaction or uneconomic with the use of a high safety factor [16].

Unlike deterministic optimization, various uncertainties are taken into account while making the best decisions in RBDO. Reliability analysis is the main tache in RBDO, where the reliability or the failure probability of performance constraint is evaluated. The probability of performance constraint $g_i(\boldsymbol{d},\boldsymbol{X}) \le 0$ can be described as

$$P(g_i(\boldsymbol{d},\boldsymbol{X}) \le 0) = \int_{g_i(\boldsymbol{X})\le 0} \cdots \int f_X(\boldsymbol{X}) d\boldsymbol{X} \tag{1}$$

Where $f_X(\boldsymbol{X})$ is the joint probability density function (PDF) of random variables.

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Generally, it's difficult to calculate the multi-dimensional numerical integration, especially for problems with high dimension or complicated failure region [1]. To efficiently and accurately calculate the probability integration, various methods including analytical and simulation methods are developed.

First-order reliability method (FORM) [5,9,28] and second-order reliability method (SORM) [10] are the most commonly used analytical methods. FORM/SORM involves linear or second-order Taylor expansion of the performance constraint at the most probable point (MPP) [32]. Analytical methods are very efficient, but it may be non-convergent for nonlinear problems or problems with complicated failure region [21]. Compared with analytical methods, the simulation methods such as Monte Carlo simulation (MCS) [26] is easy to implement, where the failure probability can be calculated for performance constraints of any form [11]. However, the computational cost of MCS is unbearable because a prohibitively large number of simulation runs are needed to calculate an accurate failure probability.

To reduce the high computational burden in MCS, various methods have been proposed, among which importance sampling (IS) method is widely accepted. By shifting the sampling center from the design point to MPP, IS chooses samples with higher failure probability [2]. However, MPP is calculated by analytical methods, which may not be accurate for complex probabilistic constraint.

In view of this, the radial-based importance sampling (RBIS) method is proposed in this paper [8]. RBIS excludes an n-dimensional sphere called "β-sphere" from the safe domain of design space and only the MCS samples outside the β-sphere are evaluated to calculate the failure probability. To determine the optimal radius of β-sphere, Frank proposed an adaptive scheme (ARBIS) using direction simulation [7]. However, it's difficult to determine the initial radius of ARBIS which makes the excluded initial sphere locate in the failure domain. To overcome this problem, a new scheme to determine the optimal radius of RBIS is needed. Because of the significance of the target reliability β^t in RBDO, it's reasonable to develop a new optimal radius calculation method for RBIS based on the target reliability β^t.

The integration strategies of reliability analysis (failure probability calculation) and design optimization also have great influence on the accuracy and efficiency of RBDO. Double-loop methods are time consuming because it involves a two-level optimization structure, where the outer optimization loop is applied to calculate the iteration design variables and the inner loop is applied to calculate the failure probability or MPP [29]. Single-loop methods [4,22] have only one optimization loop because the reliability analysis loop is approximated by its Karush–Kuhn–Tucker (KKT) optimality conditions. Decoupled-loop methods [6,12,13] separate the reliability analysis loop from the design optimization loop, then the RBDO problem is converted into a sequence of deterministic optimization problems. Moreover, after the failure probability and its gradient are calculated, sequential approximation programming (SAP) scheme can also be used to solve RBDO problems [18,24].

In this paper, to improve the efficiency of MCS-based RBDO method, a new scheme combining RBIS, arc search, line interpolation and SAP is developed. RBIS is used to calculate the failure probability and its gradient. Arc search and line interpolation are used to determine the optimal radius of RBIS and check the feasibility of probabilistic constraints. SAP is used to calculate the next design iteration point.

2. Related Work

In the following, for simplicity, random variable $\boldsymbol{X}$ is regarded as statistically independent and follows Gaussian distribution. If random variables $\boldsymbol{X}$ is dependent or follows other distribution, Rosenblatt transformation [20] or Nataf transformation [19] can be used.

2.1. Monte Carlo simulation

Monte Carlo simulation (MCS) is commonly used to calculate the failure probability and its corresponding gradient. MCS is achieved through realizing random variables and determining whether a particular event occurs for a simulation experiment [17]. Using MCS, the failure probability is estimated by counting the number of failure samples N_f and dividing it by the total number of samples N [14].

$$P_f = \frac{1}{N}\sum_{i=1}^{N} I_F(\boldsymbol{X}^i) = \frac{N_f}{N}$$
$$\text{where } I_F(\boldsymbol{X}) = \begin{cases} 1, \boldsymbol{X} \in F \\ 0, \boldsymbol{X} \notin F \end{cases} \tag{2}$$

The gradient information $\frac{\partial P_f(\boldsymbol{\mu}_X^k)}{\partial \boldsymbol{\mu}_X}$ is calculated as follows [23]:

$$\begin{aligned}\frac{\partial P_f(\boldsymbol{\mu}_X^k)}{\partial \boldsymbol{\mu}_X} &= \frac{1}{N}\sum_{i=1}^{N} \frac{I_F(\boldsymbol{X}^i)}{f_X(\boldsymbol{X}^i)} \frac{\partial f_X(\boldsymbol{X}^i)}{\partial \boldsymbol{\mu}_X} \\ &= \frac{1}{N}\sum_{i=1}^{N} \frac{I_F(\boldsymbol{X}^i)(\boldsymbol{X}^i - \boldsymbol{\mu}_{X^i})}{\sigma_{X^i}^2}\end{aligned} \tag{3}$$

Note that the simulation samples used in Equation (3) are the same as that used to calculate the failure probability $P_f(\boldsymbol{\mu}_X^k)$. In other words, calculation of $\frac{\partial P_f(\boldsymbol{\mu}_X^k)}{\partial \boldsymbol{\mu}_X}$ does not require additional simulation runs.

2.2. Arc search scheme in PMA+

Performance measure approach (PMA) is an effective method to conduct reliability analysis. In PMA, the minimum performance target point (MPTP) is the point locating on the β^t -sphere with the smallest performance function value. The calculation of MPTP is a process of arc search, which is defined as

$$\begin{aligned}&\min\ \ \mathrm{G}(\boldsymbol{U}) \\ &\text{s.t.}\ \ \ \|\boldsymbol{U}\| = \beta^t\end{aligned} \tag{4}$$

Because of its stability and efficiency, hybrid mean value (HMV) method, which adaptively selects advanced mean value (AMV) method or conjugate mean value (CMV) method according to the form of the performance function, is employed to solve the optimization problem in Equation (4) [31].

To further improve the efficiency of MPTP search, Choi et al. proposed a fast reliability analysis method in enriched performance measure approach (PMA+) [30]. Instead of the design point, MPTP obtained previously is selected as initial iteration point in PMA+. Therefore, some information obtained in the previous RBDO iteration can be reused and the efficiency of the MPTP search is improved significantly.

3. Method

Unlike MCS, where all the random samples are evaluated, RBIS only calculates the performance function value for those samples outside a β -sphere located in safe region. Therefore, it is an efficient, accurate and robust approach to calculate the failure probability and its gradient. In this paper, RBIS is selected to conduct reliability and sensitivity analysis. A new scheme combining arc search and linear interpolation is developed to calculate the optimal radius of β -sphere. To further improve the efficiency, a novel probabilistic constraint feasibility check method is developed and only the active probabilistic constraints need to conduct reliability analysis. Once the failure probability and its gradient are obtained, the next design point in RBDO is calculated by SAP.

3.1. Failure probability and gradient calculation using RBIS

RBIS is based on a mathematical theory that the sum of the squares of n independent standard normal random variables follows the chi-squared distribution with n degrees of freedom.

Therefore, excluding a β -sphere from the safe domain and sampling outside the sphere, the failure probability can be calculated as follows

$$P_f = \frac{1}{N_o}\sum_{i=1}^{N_o} I_F(\boldsymbol{X}^i) = \frac{N_{fo}}{N_o}(1-\chi_n^2(\beta^2))$$
$$I_F(\boldsymbol{X}) = \begin{cases} 1, \boldsymbol{X} \in F \\ 0, \boldsymbol{X} \notin F \end{cases} \tag{5}$$

Where N_o is the number of samples outside the β -sphere and N_{fo} is number of the failure samples. β is the radius of the sphere which is located inside the safe domain in standard normal space of RBDO.

The gradient of the failure probability can also be calculated as

$$\frac{\partial P_f(\boldsymbol{\mu}_{\boldsymbol{X}}^k)}{\partial \boldsymbol{\mu}_{\boldsymbol{X}}} = \frac{1}{N_o}\sum_{i=1}^{N_o} \frac{I_F(\boldsymbol{X}^i)(\boldsymbol{X}^i - \boldsymbol{\mu}_{\boldsymbol{X}^i})}{\sigma_{\boldsymbol{X}^i}^2}$$
$$I_F(\boldsymbol{X}) = \begin{cases} 1, \boldsymbol{X} \in F \\ 0, \boldsymbol{X} \notin F \end{cases} \tag{6}$$

Like MCS, the samples $\boldsymbol{X}^i, i = 1,\cdots,N_o$ in Equation (6) are the same as that used for estimating the failure probability $P_f(\boldsymbol{\mu}_{\boldsymbol{X}}^k)$, which means the estimation of $\frac{\partial P_f(\boldsymbol{\mu}_{\boldsymbol{X}}^k)}{\partial \boldsymbol{\mu}_{\boldsymbol{X}}}$ does not require additional MCS simulation runs.

3.2. Determination of optimal radius of RBIS

To reduce the number of performance function evaluation as far as possible, the β -sphere should be tangent to the limit state surface. Therefore, the optimal radius β is equal to the reliability index β_{RIA}, which is the shortest distance from the origin to the limit state boundary in standard normal space. The calculation of reliability index β_{RIA} is defined as [3]

$$\begin{aligned} &\min \ \beta_{\mathrm{RIA}} = \|\boldsymbol{U}\| \\ &\text{s.t.} \quad \mathrm{G}(\boldsymbol{U}) = 0 \end{aligned} \tag{7}$$

However, reliability index approach (RIA) has been demonstrated to be inefficient and unstable by many researches [15]. Therefore, a new scheme based on arc search in PMA+ and linear interpolation is proposed in this paper to determine the optimal radius of RBIS.

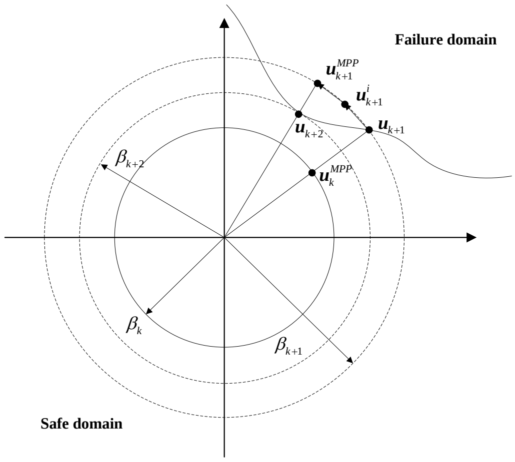


Figure 1. Optimal radius calculation using the arc search scheme in PMA+ and line interpolation

All the process for determining the optimal radius of this new scheme is conducted in U-space (standard normal space) (Figure 1). The detailed procedure is as follows:

1) Excluding a β_k -sphere in the design domain. The initial β_0 is determined as $\beta_0 = 1.5\beta^t$, which can also be applied to check the feasibility of probabilistic constraint. Detailed explanation for β_0 is lunched in Section 3.3.

2) Searching MPP on β_k -sphere using the arc search scheme in PMA+.

3) Applying the line interpolation scheme to search the point $\boldsymbol{u}_k$ on the limit state boundary using the performance function value of MPP and the current design point.

4) Constructing a new β_k -sphere using $\|\boldsymbol{u}_k\|$ as radius and searching the next MPP using arc search scheme in PMA+.

5) Repeating Step 2 to 4 until converge. Then the terminal β_k will be the optimal radius for RBIS.

3.3. Feasibility check for probabilistic constraints

In RBDO, high computational cost is spent to conduct reliability analysis and design optimization [30]. However, not all probabilistic constraints are active in design optimization phase. So it is necessary to check the feasibility status of probabilistic constraints and avoid accurate reliability analysis for inactive constraints. In this paper, a new probabilistic constraints feasibility check method is developed as:

$$\begin{aligned} &\begin{cases} g(\boldsymbol{u}_0^{MPP}) > 0, \text{inactive} \\ g(\boldsymbol{u}_0^{MPP}) < 0, \text{active} \end{cases} \\ &\text{where } \boldsymbol{u}_0^{MPP} \text{ is the optimal solution of problem:} \\ &\min g(\boldsymbol{u}) \\ &\text{s.t. } \|\boldsymbol{u}\| = 1.5\beta^t \end{aligned} \tag{8}$$

In every design optimization iteration, the initial radius of β -sphere is set as $\beta_0 = 1.5\beta^t$. Then MPP is searched on this β_0 -sphere using arc search in PMA+ method, if the performance function value $g(\boldsymbol{u}_0^{MPP}) > 0$, which means the reliability of this probabilistic constraint is larger than $1.5\beta^t$, it is identified as inactive.

Once the feasibility status of probabilistic constraints is checked, RBIS is used to calculate the failure probability and its gradient for active probabilistic constraints.

3.4. Procedures and flowchart of the proposed MORS method

The flowchart of the proposed MORS method is given in Figure 2. It consists of five procedures:

1) Initializing the radius of RBIS $\beta_0 = 1.5\beta^t$. Checking the feasibility status of probabilistic constraints using the proposed method.

2) Determining the optimal radius β_{opt} of RBIS using the proposed moving optimal radius (MORS) scheme for active probabilistic constraints.

3) Generating random samples in X-space using MCS method. Excluding the β_{opt} -sphere in U-space and evaluating the performance function value of samples outside the β_{opt} -sphere.

4) Calculating the failure probability and its gradient for active probabilistic constraints using RBIS.

5) Calculating the next design point. Based on the failure probability and its gradient obtained above, the next design point is obtained using SAP.

6) If converged, then end; otherwise, go to step (1).

4. Applications

In this section, RBDO application of a honeycomb crashworthiness design, a nonlinear numerical example and a reducer design problem are used to verify the accuracy and efficiency of proposed MORS method. MCS-SAP with 10^6 sample size in each iteration is used for comparison. All program codes are performed in "MATLAB7.11".

4.1. Honeycomb crashworthiness design application

Having outstanding potential in energy absorption, thermal isolation, dynamic and acoustic damper, Honeycomb cellular structures (Figure 3) have recently been a popular research topic [25]. This study conducted the predominantly axial crushing test for the aluminum honeycombs.

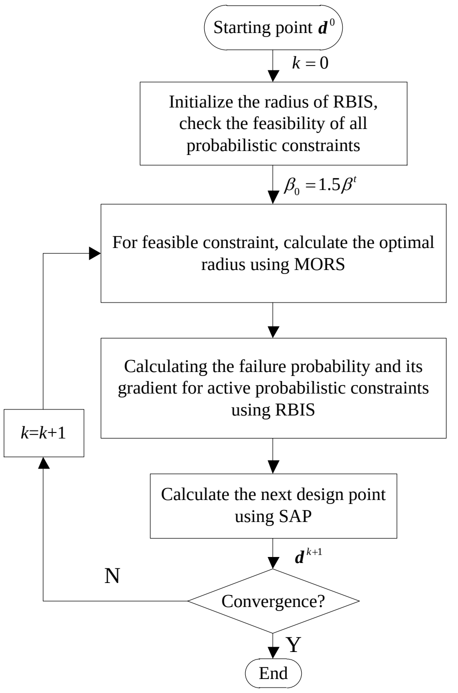


Figure 2. Flowchart of proposed method.

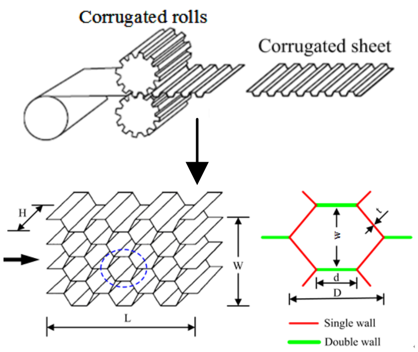


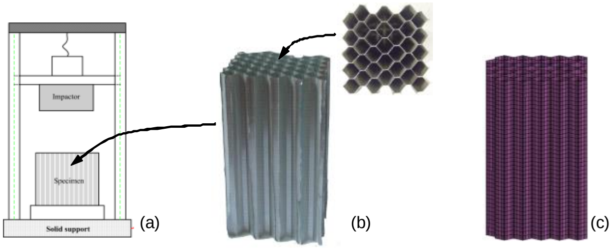
Figure 3. Nomenclature of the hexagonal honeycomb

The optimization objective is to maximize the specific energy absorption (SEA) which is defined as

$$\text{SEA} = \frac{\text{Total absorbed energy}}{\text{Total structural weight}} \tag{9}$$

Taking human or product safety into consideration, the deceleration limit $\alpha_{\max}$ during crashing process should not exceed a certain level α_{const} . In Sun’s study, the variations of factors t and σ_0 appear more sensitive to SEA and $\alpha_{\max}$ [25]. Therefore, these two parameters are chosen as the design variables in RBDO of a honeycomb material.

Nonlinear dynamic finite element analysis program LS-DYNA was used to simulate the crashworthiness process of honeycomb structure (Figure 4).



(a) Schematic of experimental setup (b) Test specimen (c) Finite element model
Figure 4. Axial impact test on Honeycomb cellular materials

The RBDO model of the honeycomb crashworthiness is described as

$$\begin{aligned}
&\text{find: } \mu_t, \mu_{\sigma_0} \\
&\text{max: } \text{SEA}(\mu_t, \mu_{\sigma_0}) \\
&\text{s.t.: } \quad \text{p}\left[\alpha_{\max} \le \alpha_{\text{const}}\right] \ge \Phi\left(\beta^t\right) \\
&t\sim N\left(\mu_t, 0.004^2\right), \sigma_0 \sim N\left(\mu_{\sigma_0}, 0.006^2\right) \\
&0.05 \le \mu_t \le 0.20, 0.15 \le \mu_{\sigma_0} \le 0.40 \\
&\beta^t = 2.0, (\mu_t^{(0)}, \mu_{\sigma_0}^{(0)}) = \left[0.05, 0.2\right]^{\text{T}}
\end{aligned} \tag{10}$$

Where (t, σ_0) are the random design variables. α_{const} is the upper limit of the acceleration, which is set 30g herein.

Table 1 displays the numerical results of MCS-SAP and MORS-SAP for the honeycomb crashworthiness design. "Sample size" is the number of performance function evaluation. β^{MCS} stands for the reliability index evaluated by MCS with 10^7 sample size. It can be seen that both the two methods converge to the same point (0.0500, 0.3325), which verifies that the proposed method has the same accuracy as MCS-SAP. Moreover, through the number of iteration is the same in the two methods, the performance function evaluation number (Sample size in Table 1) in proposed method is only 9.13% (7*e6/6.39*e5) of that in MCS-SAP, which shows its high efficiency.

Table 1. Summary of the optimization results for honeycomb crashworthiness design

Method	Design variable	Objection Function	Iteration Number	Sample size	β^{MCS}
MCS-SAP	(0.0500,0.3325)	31.3043	7	7*e6	1.998
MORS-SAP	(0.0500,0.3325)	31.3043	7	6.39*e5	1.998

The detailed iteration histories of the two methods are compared in Table 2 and Table 3. The function evaluation number in every iteration of the two methods are compared in Figure 5 (Blue Bar: MCS-SAP; Red Bar: MORS-SAP). "R" stands for the radius of excluded sphere in U-space. "Inf." means the value of β evaluated by MCS at the current design is infinite. Both the MCS-SAP and the proposed MORS-SAP method have 7 iterations. But in MCS-SAP, the failure probability and its gradient need to be evaluated at all design iterations, even when the probabilistic constraint is inactive. This causes a huge waste of computational resource in inactive performance function evaluation. The proposed MORS-SAP method can overcome this problem by checking the feasibility of probabilistic constraint first and then only conducting failure probability and its gradient calculation for active probabilistic constraints. Moreover, reliability analysis using RBIS instead of MCS disregards the performance function evaluation for the sampling points located inside the β -sphere. Therefore, the number of performance function evaluation can be significantly reduced (6.39e5 in MORS-SAP to7e6 in MCS-SAP).

Table 2. Iteration history of MCS-SAP method for honeycomb crashworthiness design

*i*th Iteration	Design Variable	Objection Function	β^{MCS}	Sample size
0	(0.0500,0.2000)	25.2786	4.215	1e6
1	(0.0500,0.2313)	26.3099	4.892	1e6
2	(0.0500,0.2625)	27.5793	Inf.	1e6
3	(0.0500,0.2938)	29.0949	3.447	1e6
4	(0.0500,0.3250)	30.8472	2.256	1e6
5	(0.0500,0.3363)	31.5413	1.840	1e6
6	(0.0500,0.3332)	31.3477	1.972	1e6
7	(0.0500,0.3325)	31.3043	1.998	\

Table 3. Iteration history of MORS-SAP method for honeycomb crashworthiness design

*i*th Iteration	Design Variable	Constraint Feasibility Status	Obj.	β^{MCS}	R	Sample size
0	(0.0500,0.2000)	Inactive	25.2786	4.215	\	\
1	(0.0500,0.2313)	Inactive	26.3099	4.892	\	\
2	(0.0500,0.2625)	Inactive	27.5793	Inf.	\	\
3	(0.0500,0.2938)	Inactive	29.0949	3.447	\	\
4	(0.0500,0.3250)	Active	30.8472	2.256	2.268	251745
5	(0.0500,0.3363)	Active	31.5413	1.840	1.850	197724
6	(0.0500,0.3332)	Active	31.3477	1.972	2.028	189808
7	(0.0500,0.3325)	\	31.3043	1.998	\	\

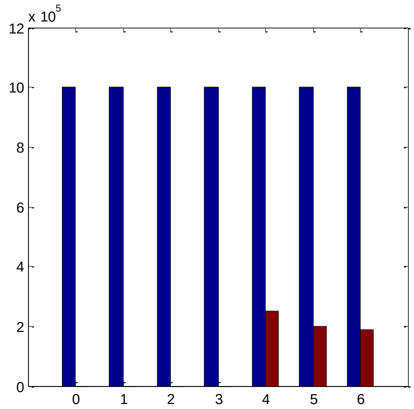


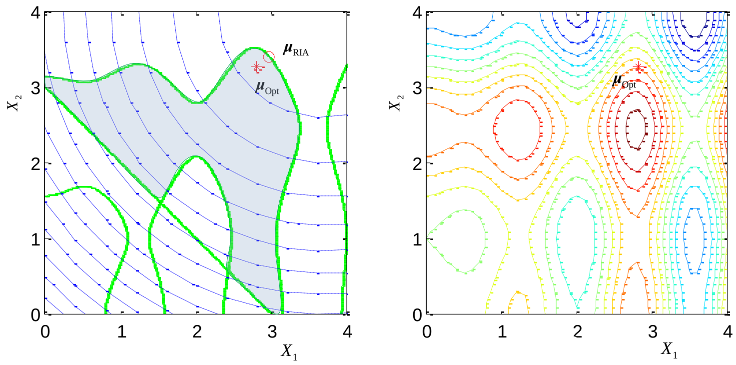
Figure 5. Function evaluation number in every iteration of the two methods in example 1

4.2. Nonlinear mathematical example

This RBDO problem is proposed by Lee and Jung [31]:

$$
\begin{aligned}
&\text{find} \quad \boldsymbol{\mu} = [\mu_1, \mu_2]^T \\
&\text{min} \quad f(\boldsymbol{\mu}) = (\mu_1 - 3.7)^2 + (\mu_2 - 4)^2 \\
&\text{s.t.} \quad P(g(\boldsymbol{X}) < 0) \le \Phi(-\beta^t) \\
&g_1(\boldsymbol{X}) = -X_1 \sin(4X_1) - 1.1 X_2 \sin(2X_2) \\
&g_2(\boldsymbol{X}) = X_1 + X_2 - 3 \\
&0.0 \le \mu_1 \le 3.7, \;\; 0.0 \le \mu_2 \le 4.0 \\
&X_i \sim N(\mu_i, 0.1^2), \; i = 1, 2 \\
&\beta_1^t = \beta_2^t = 2.0, \; \boldsymbol{\mu}^{(0)} = [2.50, 2.50]
\end{aligned}
\tag{11}
$$

As shown in Figure 6(a), the objective function of this problem is a simple quadratic function denoted with dotted lines, whose value decreases up and to the right in the design space. The shaded area is the feasible design domain and the solid lines denote the limit state constraints. In Figure 6(b), contour lines show the highly nonlinearity of the first constraint.



a Feasible design domain b Contour lines of the first constraint

Figure 6. Graphical representation of example 2

Table 4 lists the RBDO results of the nonlinear example using the two methods mentioned above.

Table 4. Summary of optimization results for example 2

Method	Design variable	Objection Function	Iteration Number	Sample size	β_1^{MCS}	β_2^{MCS}
MCS-SAP	(2.8421,3.2324)	1.3258	9	1.80*e7	2.001	Inf.
MORS-SAP	(2.8421,3.2324)	1.3258	9	1.55*e6	2.001	Inf.

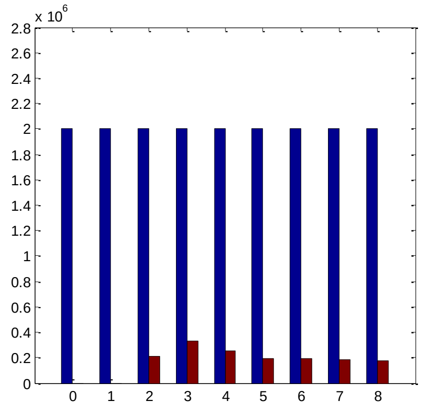


Figure 7. Function evaluation number in every iteration of the two methods in example 2

In Table 4, the number of iteration is the same in the two methods. However, the number of function call in proposed method is only 8.61% of that in MCS-SAP, which shows its high efficiency. Moreover, the result from the proposed method is identical with that from MCS-SAP, which means the proposed method is also very accurate.

Table 5. Iteration history of MCS-SAP method for example 2

*i*th Iteration	*Design Variables*	Objection Function	β_1^{MCS}	β_2^{MCS}	*Sample size*
0	(2.5000,2.5000)	3.6900	Inf.	Inf.	1e6*2
1	(2.7467,2.7667)	2.4298	5.199	Inf.	1e6*2
2	(2.9933,3.0333)	1.4339	1.800	Inf.	1e6*2
3	(2.8305,3.3000)	1.2460	1.610	Inf.	1e6*2
4	(2.8666,3.2330)	1.2828	1.811	Inf.	1e6*2
5	(2.8110,3.2764)	1.3139	1.887	Inf.	1e6*2
6	(2.8348,3.2468)	1.3159	1.954	Inf.	1e6*2
7	(2.8640,3.2106)	1.3220	1.967	Inf.	1e6*2
8	(2.8442,3.2306)	1.3244	1.994	Inf.	1e6*2
9	(2.8421,3.2320)	1.3258	2.001	Inf.	

Table 6. Iteration history of MORS-SAP method for example 2

*i*th Iteration	*Design Variables*	Constraint Feasibility Status		Objection Function	β_1^{MCS}	R_1	β_2^{MCS}	R_2	*Sample size*
		Con. 1	*Con. 2*						
0	(2.5000,2.5000)	Inactive	Inactive	3.6900	Inf.	\	Inf.	\	\
1	(2.7467,2.7667)	Inactive	Inactive	2.4298	5.199	\	Inf.	\	\
2	(2.9933,3.0333)	Active	Inactive	1.4339	1.800	1.755	Inf.	\	215098
3	(2.8305,3.3000)	Active	Inactive	1.2460	1.610	1.493	Inf.	\	327486
4	(2.8667,3.2330)	Active	Inactive	1.2827	1.811	1.661	Inf.	\	251745
5	(2.8112,3.2761)	Active	Inactive	1.3140	1.888	1.801	Inf.	\	197724
6	(2.8345,3.2469)	Active	Inactive	1.3162	1.954	1.821	Inf.	\	189808
7	(2.8600,3.2151)	Active	Inactive	1.3217	1.969	1.825	Inf.	\	188836
8	(2.8417,3.2334)	Active	Inactive	1.3244	1.993	1.854	Inf.	\	179166
9	(2.8421,3.2320)	\	\	1.3258	2.001	\	Inf.	\	\

The detailed iteration histories of the two methods are compared in Table 5 and Table 6. In every iteration, the function evaluation number of MCS (Blue Bar) and MORS (Red Bar) are compared in Figure 7. Both methods have 9 iterations, but at each design iteration the proposed MORS-SAP method has fewer performance function call than MCS-SAP. Using the proposed probabilistic constraint feasibility check scheme, probabilistic constraint 1 and 2 are regarded as inactive during the first two iterations, thus there is no need to calculate failure probability and its gradient. In the following iterations, because probabilistic constraint 1 is active and probabilistic constraint 2 is inactive, only probabilistic constraint 1 needs to conduct failure probability and gradient calculation. In the process of calculating failure probability and gradient, a β-sphere is excluded from the MCS sampling domain and only the samples outside the sphere are evaluated using the proposed method. Because most samples of MCS are generated around the current design, the efficiency of proposed MORS-SAP is much more efficient. Moreover, with the radius of the β-sphere growing, the number of samples necessary to conduct performance function evaluation is reducing.

4.3. Speed reducer design

The speed reducer design (Figure 8) is usually used to test the performance of RBDO method [27]. The RBDO model is formulated as

$$
\begin{aligned}
&\text{find}: \boldsymbol{\mu} = \left[\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7\right]^{\mathrm{T}} \\
&\min: f\left(\boldsymbol{\mu}\right) = 0.7854\mu_1\mu_2^2(3.3333\mu_3^2 + 14.9334\mu_3 \\
&-43.0934) - 1.5080\mu_1\left(\mu_6^2 + \mu_7^2\right) + 7.4770\left(\mu_6^3 + \mu_7^3\right) \\
&+0.7854(\mu_4\mu_6^2 + \mu_5\mu_7^2) \\
&\text{s.t.}: \quad \mathrm{P}\left[g_i\left(\boldsymbol{X}\right) > 0\right] \le \Phi\left(-\beta_i^t\right), i = 1, \cdots, 11 \\
&\text{where} \quad g_1\left(\boldsymbol{X}\right) = \frac{27}{X_1X_2^2X_3} - 1, g_2\left(\boldsymbol{X}\right) = \frac{397.5}{X_1X_2^2X_3^2} - 1 \\
&g_3\left(\boldsymbol{X}\right) = \frac{1.93X_4^3}{X_2X_3X_6^4} - 1, g_4\left(\boldsymbol{X}\right) = \frac{1.93X_5^3}{X_2X_3X_7^4} - 1 \\
&g_5\left(\boldsymbol{X}\right) = \frac{\sqrt{(745X_4/(X_2X_3))^2 + 16.9\times10^6}}{0.1X_6^3} - 1100 \\
&g_6\left(\boldsymbol{X}\right) = \frac{\sqrt{(745X_5/(X_2X_3))^2 + 157.5\times10^6}}{0.1X_7^3} - 850 \\
&g_7\left(\boldsymbol{X}\right) = X_2X_3 - 40, g_8\left(\boldsymbol{X}\right) = 5 - \frac{X_1}{X_2} \\
&g_9\left(\boldsymbol{X}\right) = \frac{X_1}{X_1} - 12, g_{10}\left(\boldsymbol{X}\right) = \frac{1.5X_6 + 1.9}{X_4} - 1 \\
&g_{11}\left(\boldsymbol{X}\right) = \frac{1.1X_7 + 1.9}{X_5} - 1, X_j \sim N\left(d_j, 0.005^2\right), j = 1, \cdots, 7 \\
&2.6 \le \mu_1 \le 3.6, 0.7 \le \mu_2 \le 0.8, 17 \le \mu_3 \le 28 \\
&7.3 \le \mu_4 \le 8.3, 7.3 \le \mu_5 \le 8.3, 2.9 \le \mu_6 \le 3.9 \\
&5.0 \le \mu_7 \le 5.5, \ \beta_i^t = 3.0, i = 1, \cdots, 11 \\
&\boldsymbol{\mu}^{(0)} = \left[3.50, 0.70, 17.00, 7.30, 7.72, 3.35, 5.29\right]^{\mathrm{T}}
\end{aligned}
\tag{12}
$$

The comparison results are shown in Tables 7 and 8. The optimal designs of the two methods are identical, which verifies that the proposed MORS-SAP is very accurate. Though there are 8 iterations in the two methods, the number of performance function call in the proposed method is only 21.58% of that in MCS, which shows the proposed method's high efficiency.

Tables 7. Summary of the optimization results for speed reducer design

Method	Design variable	Objection Function	Iteration Number	*Sample size*
MCS-SAP	(3.5765,0.7000,17.0000, 7.3000,7.7542,3.3652,5.3017)	3038.6361	8	8.8e7
MORS-SAP	(3.5765,0.7000,17.0000, 7.3000,7.7542,3.3652,5.3017)	3038.6361	8	1.899e7

Tables 8. Summary of MCS results for speed reducer design

Method	β_1^{MCS}	β_2^{MCS}	β_3^{MCS}	β_4^{MCS}	β_5^{MCS}	β_6^{MCS}	β_7^{MCS}	β_8^{MCS}	β_9^{MCS}	$\beta_{10}^{\mathrm{MCS}}$	$\beta_{11}^{\mathrm{MCS}}$
MCS-SAP	Inf.	Inf.	Inf.	Inf.	3.000	3.001	Inf.	3.002	Inf.	Inf.	3.005
MORS-SAP	Inf.	Inf.	Inf.	Inf.	3.000	3.001	Inf.	3.002	Inf.	Inf.	3.005

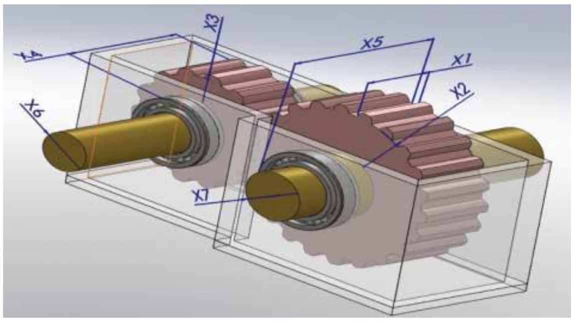


Figure 8. The speed reducer

The detailed iteration histories of MCS-SAP and proposed MORS-SAP method are compared in Table 9-12. In every design iteration, the function evaluation number of the two methods is compared in Figure 9. In all the RBDO iterations, probabilistic constraint 1-4, 7 and 9-10 are inactive, thus there is no need to conduct failure probability and its gradient calculation for these probabilistic constraints in proposed MORS-SAP method. For other active probabilistic constraints, because only the samples outside the β-sphere are evaluated, the sample size of these probabilistic constraints is much less than that of MCS-SAP. Therefore, the proposed MORS-SAP is accurate and more efficient.

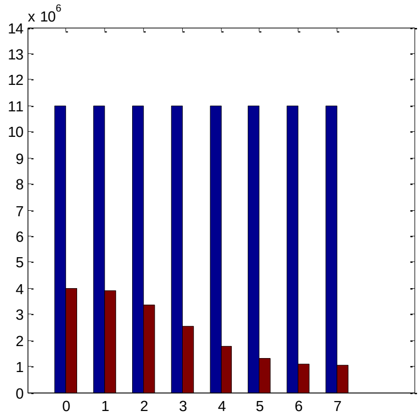


Figure 9. Function evaluation number in every iteration of the two methods in example 3

Table 9. Iteration history of MCS-SAP method for speed reducer design

*i*th Iteration	Design Variables	Objection Function	Sample size
0	(3.5000,0.7000,17.0000,7.3000,7.7200,3.3500,5.2900)	2996.5158	1.1e7
1	(3.5319,0.7000,17.0000,7.3000,7.7325,3.3565,5.2938)	3013.3927	1.1e7
2	(3.5463,0.7000,17.0000,7.3000,7.7396,3.3594,5.2964)	3021.5987	1.1e7
3	(3.5572,0.7000,17.0000,7.3000,7.7450,3.3615,5.2985)	3027.8713	1.1e7
4	(3.5658,0.7000,17.0000,7.3000,7.7493,3.3633,5.3001)	3032.8220	1.1e7
5	(3.5722,0.7000,17.0000,7.3000,7.7523,3.3644,5.3011)	3036.3191	1.1e7
6	(3.5756,0.7000,17.0000,7.3000,7.7537,3.3650,5.3015)	3038.0930	1.1e7
7	(3.5764,0.7000,17.0000,7.3000,7.7541,3.3652,5.3017)	3038.5947	1.1e7
8	(3.5765,0.7000,17.0000,7.3000,7.7542,3.3652,5.3017)	3038.6361	\

Table 10. MCS results of the MCS-SAP method for speed reducer design

*i*th Iteration	β_1^{MCS}	β_2^{MCS}	β_3^{MCS}	β_4^{MCS}	β_5^{MCS}	β_6^{MCS}	β_7^{MCS}	β_8^{MCS}	β_9^{MCS}	β_{10}^{MCS}	β_{11}^{MCS}
0	Inf.	Inf.	Inf.	Inf.	-0.042	0.668	Inf.	0	Inf.	Inf.	0.135
1	Inf.	Inf.	Inf.	Inf.	1.258	1.425	Inf.	1.253	Inf.	Inf.	1.254
2	Inf.	Inf.	Inf.	Inf.	1.837	1.945	Inf.	1.816	Inf.	Inf.	1.824
3	Inf.	Inf.	Inf.	Inf.	2.258	2.364	Inf.	2.246	Inf.	Inf.	2.236
4	Inf.	Inf.	Inf.	Inf.	2.622	2.685	Inf.	2.585	Inf.	Inf.	2.584
5	Inf.	Inf.	Inf.	Inf.	2.832	2.887	Inf.	2.824	Inf.	Inf.	2.840
6	Inf.	Inf.	Inf.	Inf.	2.962	2.972	Inf.	2.971	Inf.	Inf.	2.967
7	Inf.	Inf.	Inf.	Inf.	2.993	3.016	Inf.	2.992	Inf.	Inf.	2.987
8	Inf.	Inf.	Inf.	Inf.	3.000	3.001	Inf.	3.002	Inf.	Inf.	3.005

Table 11. Iteration history of MORS-SAP method for speed reducer design

*i*th Iteration	*Design Variables*	Objection Function	R_5	R_6	R_8	R_{11}	*Sample size*
0	(3.5000,0.7000,17.0000,7.3000, 7.7200,3.3500,5.2900)	2996.5158	0	0.664	0	0.134	4e6
1	(3.5322,0.7000,17.0000,7.3000, 7.7327,3.3565,5.2939)	3013.5785	1.254	1.444	1.263	1.255	3.893e6
2	(3.5465,0.7000,17.0000,7.3000, 7.7396,3.3593,5.2965)	3021.7153	1.809	1.955	1.824	1.815	3.367e6
3	(3.5575,0.7000,17.0000,7.3000, 7.7451,3.3615,5.2986)	3028.0549	2.247	2.375	2.254	2.244	2.542e6
4	(3.5663,0.7000,17.0000,7.3000, 7.7494,3.3632,5.3002)	3033.0586	2.588	2.690	2.602	2.585	1.782e6
5	(3.5723,0.7000,17.0000,7.3000, 7.7524,3.3644,5.3012)	3036.4244	2.822	2.895	2.837	2.833	1.296e6
6	(3.5757,0.7000,17.0000,7.3000, 7.7537,3.3650,5.3016)	3038.1961	2.938	2.978	2.970	2.955	1.081e6
7	(3.5766,0.7000,17.0000,7.3000, 7.7540,3.3652,5.3016)	3038.6071	2.976	2.987	3.006	2.988	1.029e6
8	(3.5765,0.7000,17.0000,7.3000, 7.7542,3.3652,5.3017)	3038.6361					\

Table 12. MCS results of the MORS-SAP method for speed reducer design

*i*th Iteration	β_1^{MCS}	β_2^{MCS}	β_3^{MCS}	β_4^{MCS}	β_5^{MCS}	β_6^{MCS}	β_7^{MCS}	β_8^{MCS}	β_9^{MCS}	$\beta_{10}^{\mathrm{MCS}}$	$\beta_{11}^{\mathrm{MCS}}$
0	Inf.	Inf.	Inf.	Inf.	-0.042	0.668	Inf.	0	Inf.	Inf.	0.135
1	Inf.	Inf.	Inf.	Inf.	1.257	1.448	Inf.	1.263	Inf.	Inf.	1.266
2	Inf.	Inf.	Inf.	Inf.	1.816	1.968	Inf.	1.826	Inf.	Inf.	1.806
3	Inf.	Inf.	Inf.	Inf.	2.249	2.384	Inf.	2.257	Inf.	Inf.	2.235
4	Inf.	Inf.	Inf.	Inf.	2.599	2.710	Inf.	2.597	Inf.	Inf.	2.586
5	Inf.	Inf.	Inf.	Inf.	2.834	2.910	Inf.	2.833	Inf.	Inf.	2.832
6	Inf.	Inf.	Inf.	Inf.	2.952	2.992	Inf.	2.968	Inf.	Inf.	2.949
7	Inf.	Inf.	Inf.	Inf.	2.991	2.985	Inf.	3.010	Inf.	Inf.	2.987
8	Inf.	Inf.	Inf.	Inf.	3.000	3.001	Inf.	3.002	Inf.	Inf.	3.005

5. **Conclusions**

In this paper, the MORS-SAP method is proposed to substantially improve the computational efficiency of MCS-SAP method while maintaining numerical accuracy and stability. In MORS-SAP, the failure probability and its gradient are calculated using RBIS scheme, which excludes a β -sphere from the safe domain and only the samples located outside the β -sphere are evaluated. A new scheme is developed to determine the optimal radius of RBIS. According to the target reliability, the initial radius for RBIS is determined, which can also be used to check the feasibility status of probabilistic constraints afterwards. For active probabilistic constraints, the radius is moving towards to the optimal one using the proposed MORS scheme which combines the arc search scheme in PMA+ and line interpolation scheme. Then the optimal design is calculated using proposed MORS- SAP method. The RBDO application of the honeycomb crashworthiness design, a nonlinear mathematical problem and the speed reducer design example are tested to verify the computational performance of proposed MORS method. Through comparison results, it can be seen that the proposed MORS method is very efficient and accurate. Also, because the proposed MORS method has the same accuracy as MCS, it can be used as the benchmarking in failure probability calculation using other methods.

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