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T-Stability of the Euler-Maruyama Algorithm for the Generalized Black-Scholes Model with Fractional Brownian Motion

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Abstract

On account of the fact that a fractional Brownian motion (fBm) with the Hurst parameter $H \in (0,1/2) \cup (1/2,1)$ cannot follow the laws of the semimartingale and the Markov process, little work is presented about the T-stability for stochastic differential equations (SDEs) with fBm. Here, three results are obtained for the generalized Black-Scholes model (SDE) with $H \in (1/3,1/2)$. Firstly, the sufficient conditions of the stochastical and asymptotical stability in the large for such equation are presented by the aid of the Lyapunov exponent. Secondly, the Euler-Maruyama (EM) numerical algorithm with a given step-size for such model is constructed. Lastly, by taking advantage of the stable average function, the sufficient conditions of the T-stability that originated from the EM algorithm are presented. All the results show that on the basis of the stability of such equation, the T-stable region produced by the EM algorithm can be found. Moreover, one numerical example is afforded to the main conclusions.

Keywords: stochastic differential equations; the generalized Black-Scholes model; fractional brownian motion; the Euler-Maruyama algorithm; T-stability

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1. Introduction

The fBm with $H \in (0,1/2) \cup (1/2,1)$ (see [23,9,16,6]), which possesses self-similarity measure and the full-term memory, provides better modeling and description of a lot of natural and social phenomena, compared to Brownian motion ($H=1/2$). However, the research of SDEs with fBm with $H \in (0,1/2) \cup (1/2,1)$, which are neither semimartingale nor Markov processes, is just the beginning (see [1,4,20,11]). Many important properties of such equations such as the T-stability and their numerical methods are worth to be researched. The following sections give a summary of the relevant completed work and brief of what I am doing.

By broadening the Hurst parameter from $H=1/2$ to $H=1$ (see [12,18]), the Black-Scholes financial model in [5,25] was extended to generalized Black-Scholes model, which can express the properties of long-range dependence and heavy tailed distribution of financial data. In [19], for linear equations with $H \in (0,1/2)$, the properties of the existence and the uniqueness of the continuous solution are given. The existence and uniqueness to the equations with $H \in (1/2,1)$ and $H \in (0,1)$ were discussed in [24] and [14,22] respectively. For the multidimensional and time-dependent SDEs with $H \in (1/2,1)$, the properties of the solutions, which are existent and exclusive, were proved in [15] by using the fractional and the Ito integral formulas. In [13], a method structured by the random matrix with the arithmetic operators and the impulse function with $H \in (1/2,1)$, was efficiently proposed. To semilinear impulsive stochastic functional differential equations with $H \in (0,1)$, the existence of mild solutions was proved in [3].

According to what I know, few results [10,28] about stability, which are significant and widely used, were given to stochastic processes with fBm. In [14], to semilinear SDEs with $H \in (0,1)$, the dissipativeness and their drift-implicit Euler method were discussed. In [22], via an auxiliary function, the exponential stability of semilinear SDEs with $H \in (1/2,1)$ was given. By means of the Lyapunov exponents, necessary and sufficient conditions about the above two stabilities were

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discussed for the generalized Black-Scholes model with $H\in(0,1)$ in [31]. In [32], for some nonlinear SDEs, by means of the Lyapunov functions, the sufficient conditions, not only of the probatilistic stability, but also of the stability in moment exponent, were established.

Moreover, to some special fractional equations in [26,14,17,21,29], the convergence of the numerical methods have also been focused on. In [27], for SDEs with $H\in(1/2,1)$, the optimal convergent rate in mean square of the general numerical methods with an equidistant stepsize has been derived. In [2], the numerical method was given by using wavelet approximation of multifractional Brownian motion.

So far, the results, which are on the random stability in almost sure sense and the random stability in moment sense for stochastic systems, are far more than T-stability (see [8]). It only requires a small number of samples to implement. My primary purpose is to fill the gap in the T-stability with the Euler-Maruyama Algorithm for generalized Black-Scholes model with fBm with $H\in(1/3,1/2)$. Furthermore, sufficient conditions of the T-stability are presented by means of the average stability function (section 3). To express these conditions, the stochastical and asymptotical stability in the large are given to such equations, as shown in section 3.

2. Preliminary

In this whole paper, let (Ω, F, P) be a space with the complete probatilistic filter flow and satisfying standard conditions. The Euclidean norm of a in R^d , $d\in N$ is defined by $|a|$.

The following d-dimensional generalized Black-Scholes model (SDE) with $H\in(1/3,1/2)$, which is produced by fBm, is concerned in my paper

$$dy(t)=\lambda y(t)dt+\mu y(t)dB^H(t),\quad t\geq 0 \tag{1}$$

where the beginning value $y_0\in R^d$ and $\lambda,\mu\in R$. The fBm with independent scalar components is on the filtered probability space and the Hurst parameter is defined by $H:=\min\{H_j\in(1/3,1/2)|\, j=1,\dots,M\}$.

The EM algorithm with the given step width $h\in(0,1)$ is applied to the equation (1)

$$Y_{n+1}=Y_n+\lambda hY_n+\mu Y_n\Delta B_n^H,\ n=1,2,\cdots \tag{2}$$

where $Y_n\approx y(t_n)$, $t_n=nh$, $n=1,2,\dots$, $Y(0)=y_0$. According to the properties of the fBm-increments in [32], the independent increments

$$\Delta B_n^H=B^H(t_{n+1})-B^H(t_n):=\sum_{j=1}^{M}\Delta B_n^{H_j} \tag{3}$$

obey the normal distribution $N(0,h^{2H})$, where

$$\Delta B_n^{H_j}=B^{H_j}(t_{n+1})-B^{H_j}(t_n),\quad j=1,\cdots,M. \tag{4}$$

Based on [19,11,29], equation (1) obtains the existence and uniqueness of its solutions. Moreover, in the interval $[0,\infty)$, the EM numerical algorithm (2) is strongly convergent with rank $2H$ in the light of [32].

Before deducing the T-stable region of the EM numerical algorithm (2), stochastical and asymptotical stability in the large of (1) is discussed first. Three definitions about stability (see [7]) are consequently introduced.

Definition 1 The SDE (1) is named as stability in probability, if for arbitrary sufficient small $c\in(0,1)$, a positive constant $d=d(c)>0$ can be found such that for $|y_0|<d$, the probability of the limit of $y(t)$ which is equal to 0 when t tends to infinity, is equal or greater than $1-c$ in almost sure.

Definition 2 The SDE (1) is named as stochastical and asymptotical stability in large, if (1) is in probatilistic stability and for all $y_0\in R$, the probability of the limit of $y(t)$ which is equal to 0 when t tends to infinity, is equal to 1 in almost sure sense.

Definition 3 The EM algorithm (2) of the SDE (1) is named to be T-stable, if (1) is stochastical and asymptotical stability in large and the solutions Y_k of (2) satisfy

$$\lim_{k\to\infty} |Y_k| = 0. \tag{5}$$

3. Region of T-stability

In the following, the T-stable region, which is produced by the EM algorithm (2) for (1), is considered. Therefore, the stochastical and asymptotical stability in the large of (1) is given by means of the Lyapunov exponent.

Theorem 1 If $\lambda<0$, the equation (1) is stochastical and asymptotical stability in the large.

Proof: According to [30], we obtain

$$y(t) = y_0 \exp(\lambda t - \frac{\mu^2}{2} t^{2H} + \mu B^H(t)) \tag{6}$$

where $t \geq 0$. Recalling $H \in (1/3,1/2)$ and the result (Lemma 2.2 in [31])

$$\lim_{t\to\infty} \frac{B^H(t)}{t} = 0, \ \ a.s. \tag{7}$$

and we can have the Lyapunov exponent

$$\limsup_{t\to\infty} \frac{1}{t} \log |y(t)| = \lambda - \frac{\mu^2}{2} \lim_{t\to\infty} t^{2H\text{-}1} + \lim_{t\to\infty} \frac{B^H(t)}{t} = \lambda \tag{8}$$

which means that the equation (1) is stochastical and asymptotical stability in the large.

In order to analyze the T-stable region, we firstly deduce its average stability function. Here, the normal distribution ΔB^H_n is taken as $U_n h^H$ whose probabilistic distribution is described as

$$\mathrm{P}(U_n = \pm 1) = \frac{1}{2}. \tag{9}$$

From (2), we have

$$\begin{aligned} Y_{n+1} &= Y_n + \lambda h Y_n + \mu U_n h^H Y_n \\ &= (1 + \lambda h + \mu U_n h^H) Y_n \end{aligned} \tag{10}$$

The average stability function is obtained by

$$R(h;\lambda,\mu) = 1 + \lambda h + \mu U_n h^H. \tag{11}$$

Making use of (10) and (11), we can get

$$R^2(h;\lambda,\mu) = (1 + \lambda h + \mu h^H)(1 + \lambda h - \mu h^H) \tag{12}$$

According to definition 3, the EM algorithm (2) of (1) is T-stable if and only if the average stability function (11) satisfies $|R(h;\lambda,\mu)|<1$. Therefore, the T-stable region is obtained as

$$H(h) = \{(h,\lambda,\mu) \,|\, (1+\lambda h)^2 - \mu^2 h^{2H} < 1\}. \tag{13}$$

In the following theorem, we try to look for appropriate h to satisfy the region (12).

Theorem 2 The EM algorithm (2) for (1) is T-stable if h, λ and μ are in one of the following sets.

(i) $$H_1(h) = \{(h,\lambda,\mu) \,|\, \lambda < 0, \lambda^2 + 2\lambda > 0, \mu^2 \leq \lambda^2, \Delta_1 < h < \Delta_2\}. \tag{14}$$

where

$$\Delta_1 = \max\{0, (\frac{\lambda^2-\mu^2}{-2\lambda})^{\frac{1}{1-2H}}\}, \quad \Delta_2 = \min\{1, (\frac{\mu^2}{\lambda^2+2\lambda})^{\frac{1}{1-2H}}\}. \tag{15}$$

(ii)

$$H_2(h) = \{(h,\lambda,\mu) \mid \lambda<0, \lambda^2+2\lambda<\mu^2, h=1\}. \tag{16}$$

(iii)

$$H_3(h) = \{(h,\lambda,\mu) \mid \lambda<0, 1<h<(\frac{\mu^2-2\lambda}{\lambda^2})^{\frac{1}{2-2H}}\}. \tag{17}$$

Proof: (i) In $H_1(h)$, the condition $\Delta_1 < h < 1, \lambda < 0, \mu^2 < \lambda^2$ can be reduced to

$$h^2 < h < h^{2H} \tag{18}$$

and

$$h^{2H}(\lambda^2-\mu^2) < (-2\lambda)h \tag{19}$$

which lead to

$$\lambda^2 h^2 < \lambda^2 h^{2H} < \mu^2 h^{2H} + (-2\lambda)h \tag{20}$$

From (20), I can have

$$1+\lambda^2h^2+2\lambda h-\lambda^2 h^{2H} < 1 \tag{21}$$

Therefore, such stepsize h satisfies the region of T-stability (13). From the condition $0<h<\Delta_2$ and $h \in H_1(h)$, we can obtain that

$$\lambda^2h^2 < \lambda^2 h < \mu^2 h^{2H} + (-2\lambda)h \tag{22}$$

which means that such h satisfies the T-stable region (13).

(ii) For any $h \in H_2(h)$, I can have

$$1+\lambda^2\cdot 1^2 + 2\lambda\cdot 1 - \mu^2\cdot 1^{2H} < 1 \tag{23}$$

which gives $h \in H(h)$.

(iii) $H_3(h)$ follows from the condition

$$\lambda<0,\ 1<h<(\frac{\mu^2-2\lambda}{\lambda^2})^{\frac{1}{2-2H}} \tag{24}$$

that

$$h^{2H} < h \tag{25}$$

and

$$\lambda^2h^2 < \mu^2h^{2H} + (-2\lambda)h^{2H} \tag{26}$$

which can give

$$\lambda^2h^2 < \mu^2h^{2H} + (-2\lambda)h \tag{27}$$

Thus, we can get $h \in H(h)$.

4. Numerical Example

Here, the dynamic relation between the parameters λ, μ , h and the T-stability of the EM algorithm (2) to (1) is shown.

Firstly, one group of the parameters

$$\lambda = -3, \quad \mu = 2, \quad x_0 = 1, \quad H = \frac{3}{8} \tag{28}$$

is given. Then ,their regions of T-stability can be computed as

$$\begin{aligned} &H_1(h_k) = \{h_k \mid 0.4823 < h < 1\}, \\ &H_2(h_k) = \{h_k \mid h=1\}, \\ &H_3(h_k) = \{h_k \mid 1 < h < 1.088\} \end{aligned} \tag{29}$$

from Theorem 2.

Secondly, we choose the parameters for the EM algorithm (2) and choose the stepsizes inside and outside the regions of the T-stability H_1 - H_3 . Fig.1 is given by Matlab to describe the choosings.

The observations from Fig.1 indicate that the numerical example is essentially in agreement with the results of Theorem 2 in my paper.

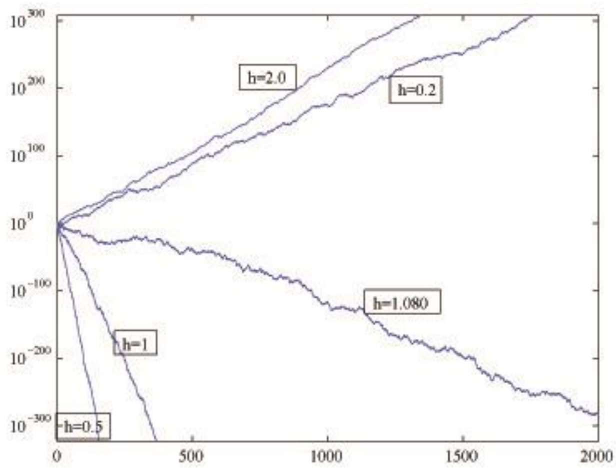

Figure 1. T-stability of the EM algorithm

5. **Conclusions**

Compared with the results of other literature, the conditions of T-stability for the generalized Black-Scholes model (SDE) with $H \in (1/3,1/2)$ and its EM algorithm are given respectively.

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